A STUDY OF CUBIC FUNCTIONS

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Abstract. We study real cubic functions using calculus techniques and use differential equation techniques to find formulas to solve cubic equations. Our analysis produces formulas that are applicable in different cases. We study the geometrical meaning of each case. Formulas are obtained in terms of radicals, circular, and hyperbolic functions. In addition we study the geometry of the complex number solutions of a cubic equation.

1. Introduction

In this paper we study a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$ with the aim to understand its geometry as well as study its roots. We assume that a, b, c, and d are real numbers and study cubic functions with real domain and codomain, and we will do so using Calculus techniques.

In order to simplify our analysis we will assume throughout this paper that $a > 0$.

As a consequence of the techniques used in the analysis of cubic functions we deduce formulas for finding the real solutions of a cubic equation. In the case that there are three solutions we obtain a formula that gives the solutions of the equation from smallest to biggest. This is similar to a formula found by François Viete, but uses sine and inverse sine instead of cosine and inverse cosine. Other formulas found for solving the cubic equation in the case that there is only one solution closely resemble other formulas by G. C. Holmes that use hyperbolic sines and cosines and their inverses. A summary of the formulas can be found in figure 2. For completeness we show how to find the real solutions of a cubic equation on every case.

This paper is organized as follows. In section 2 we collect a few geometric facts about graphs of cubic functions. These are important because each of the formulas that solve the cubic equation contains terms that have some geometric meaning in regards to its graph. In section 3 we turn our attention to the problem of solving the cubic equation. The method we use consists in analyzing level sets of the graph of a cubic. We obtain a differential equation for each solution of each level set, given by (16). Solving this equation is done by analyzing different cases as is done step by step in that section. In section 4 we deduce equivalent radical expressions for some of the formulas we deduced in section 3. Section 5 shows extra facts about graphs that expand on facts about graphs of cubic functions discussed in earlier sections. In particular we notice that changes of variables used to compute integrals have an algebraic meaning that leads to algebraic techniques to solve cubic equations. Section 6 shows the deduction of the complex solutions of a cubic equation with real coefficients. In section 7 we write those solutions in radical form. Finally in section 8 we will discuss how to remove the restriction that $a > 0$ and write final formulas to solve the cubic equation with real coefficients.

2. Cubic Functions

When graphing a cubic function $f(x) = ax^3 + bx^2 + cx + d$ we find the critical numbers of f by solving the equation $f'(x) = 0$. This leads us to the equation $3ax^2 + 2bx + c = 0$. The number of real solutions of this equation is determined by its discriminant equal to $D = (2b)^2 - 4(3a)(c) = 4(b^2 - 3ac)$. The following table determines the number of critical points

$b^2 - 3ac > 0$	There are two critical points x_1 and x_2, where $x_1 < x_2$. f is increasing in the intervals $(-\infty, x_1]$ and $[x_2, \infty)$, and f is decreasing in the interval $[x_1, x_2]$.
$b^2 - 3ac = 0$	There is only one critical point. The cubic function is increasing.
$b^2 - 3ac < 0$	There are no real critical points. The cubic function is increasing.

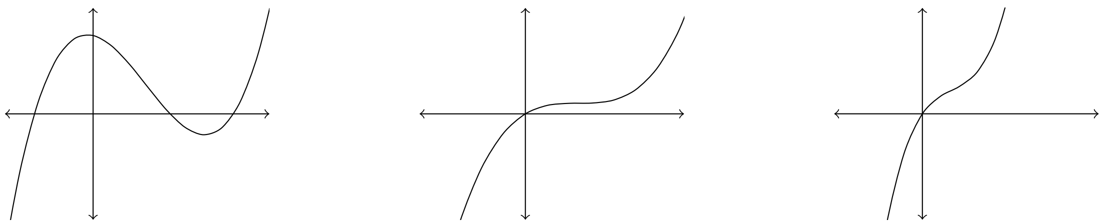
In the case that $b^2 - 3ac > 0$ the specific formulas that determine x_1 and x_2 are

$$x_1 = \frac{-b - \sqrt{b^2 - 3ac}}{3a},$$

and

$$x_2 = \frac{-b + \sqrt{b^2 - 3ac}}{3a}.$$

As a consequence of this analysis, the shape of the graph of a cubic equation is approximately the one given in Figure 1.



case $b^2 - 3ac > 0$ case $b^2 - 3ac = 0$ case $b^2 - 3ac < 0$

FIGURE 1. Possible shapes of the graph of a cubic function

Observe that the number d does not affect the shape of the graph. It only affects its position. In the case that $b^2 - 3ac > 0$, different values of d produce graphs with the same shape that are vertical translations of each other. By changing the value of d we can create functions that have 3 real zeros, 2 different real zeros, or just one real zero. In the other cases where $b^2 - 3ac \leq 0$ a cubic function only has one real zero.

The quantity that helps us determine the number of real zeros in a cubic function is its discriminant. This is defined as

$$\Delta_3(f) = a^4(\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2,$$

where α, β, and γ are the complex zeros of f. In the case that $\Delta_3(f) > 0$ f has three real roots, if $\Delta_3(f) = 0$ there is a repeated root, and if $\Delta_3(f) < 0$ there is only one real root. It can be shown that the discriminant of a cubic funtion can be computed from its coefficients by the formula

$$\Delta_3(f) = b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2. \tag{1}$$

In our computations in the future sometimes it will be necessary to resort to some useful identities, based on the identity given in the following

Lemma 1. *Let a, b, c, and d four numbers, and let $f(x) = ax^3 + bx^2 + cx + d$, then*

$$27a^2\Delta_3(f) = 4(b^2 - 3ac)^3 - (27a^2d + 2b^3 - 9abc)^2. \tag{2}$$

Proof. Let us expand f as a Taylor polynomial, then

$$f(x) = f(h) + f'(h)(x - h) + \frac{f''(h)}{2}(x - h)^2 + \frac{f'''(h)}{6}(x - h)^3,$$

for any number h. If we choose h so that $f''(h) = 0$, that is $h = -\frac{b}{3a}$, then

$$f(x) = f(-\frac{b}{3a}) + f'(-\frac{b}{3a})\left(x + \frac{b}{3a}\right) + a\left(x + \frac{b}{3a}\right)^3.$$

It follows that the polynomial

$$g(x) = f(-\frac{b}{3a}) + f'(-\frac{b}{3a})x + ax^3,$$

is obtained as a horizontal translation of the polynomial f, so it has the same discriminant as f, therefore,

$$\Delta_3(f) = \Delta_3(g) = -4af'(-\frac{b}{3a})^3 - 27a^2f(-\frac{b}{3a})^2.$$

Given that

$$f(-\frac{b}{3a}) = \frac{27a^2d + 2b^3 - 9abc}{27a^2}, \tag{3}$$

$$f'(-\frac{b}{3a}) = -\frac{b^2 - 3ac}{3a}, \tag{4}$$

the proof follows by direct substitution of these values into the previous equation. □

An important particular case of this Lemma is the case when $d = 0$. In that case we get the equality

$$27a^2(b^2c^2 - 4ac^3) = 4(b^2 - 3ac)^3 - (2b^3 - 9abc)^2.$$

which implies the equality

$$27a^2(b^2c^2 - 4ac^3) + (2b^3 - 9abc)^2 = 4(b^2 - 3ac)^3. \tag{5}$$

As a corollary, we obtain the following upper bound for the discriminant

Corollary 1.1. *Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic function, where a, b, c, and d are real numbers, then*

$$\Delta_3(f) \leq \frac{4(b^2 - 3ac)^3}{27a^2}. \tag{6}$$

Equality only holds when

$$d = \frac{9abc - 2b^3}{27a^2}. \tag{7}$$

Equality (7) only holds when $27a^2d + 2b^3 - 9abc = 0$, that is, because of equation (3), when $f(-\frac{b}{3a}) = 0$. Another important corollary of the above inequality is the following

Corollary 1.2. *Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic function, where a, b, c, and d are real numbers, then*

(1) *If $b^2 - 3ac < 0$, the function has only one real root.*
(2) *If $\Delta_3(f) > 0$, then $b^2 - 3ac > 0$.*

Proof. To prove the Corollary, let us notice the following

(1) If $b^2 - 3ac < 0$, then by inequality (6) $\Delta_3(f) < 0$, so that the equation $f(x) = 0$ has only one real solution.
(2) Assume that $\Delta_3(f) > 0$ for some numbers a, b, c, and d, then by inequality (6) we deduce that $(b^2 - 3ac)^3 > 0$, so that $b^2 - 3ac > 0$.

□

Remark 1.1. *Recall that a quadratic polynomial $q(x) = ax^2 + bx + c$ also has a discriminant, which we will also denote by $\Delta_2(q)$, but it is defined as $\Delta_2(q) = a^2(\beta - \gamma)^2$, where β and γ are the roots of q. Alternatively we can use the formula $\Delta_2(q) = b^2 - 4ac$.*

Theorem 2. *Let a, b, c and d be real numbers and $f(x) = ax^3 + bx^2 + cx + d$. Assume that α is a root of f, then*

$$\Delta_3(f) = f'(\alpha)^2\Delta_2(\frac{f}{x - \alpha}). \tag{8}$$

See footnote[1].

[1]This equality can be used to deduce equation (1) by noticing that $f'(\alpha) = 3a\alpha^2 + 2b\alpha + c$ and $\Delta_2(\frac{f}{x-\alpha}) = b^2 - 4ac - 2ab\alpha - 3a^2\alpha^2$, and then noticing that by polynomial long division we get $(3a\alpha^2 + 2b\alpha + c)^2(b^2 - 4ac - 2ab\alpha - 3a^2\alpha^2) = (27a^2d^2 + 4b^3 - 18abc - 27a^2c\alpha - 27a^2b\alpha^2 - 27a^3\alpha^3)f(\alpha) + b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2$.

Proof. Assume that the roots of f are α, β, and γ, then

$$f(x) = a(x-\alpha)(x-\beta)(x-\gamma),$$

so that

$$f'(\alpha) = a(\alpha-\beta)(\alpha-\gamma).$$

In addition

$$\Delta_2(\frac{f}{x-\alpha}) = \Delta_2(a(x-\beta)(x-\gamma)) = a^2(\beta-\gamma)^2.$$

It follows that

$$\Delta_3(f) = a^4(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 = (a^2(\alpha-\beta)^2(\alpha-\gamma)^2)(a^2(\beta-\gamma)^2) = f'(\alpha)^2\Delta_2(\frac{f}{x-\alpha})$$

□

Remark 2.1. *The generalization of equation (8) to polynomials of any degree is actually correct, that is, we always have*

$$\Delta_n(f) = f'(\alpha_1)^2\Delta_{n-1}(\frac{f}{x-\alpha_1}),$$

where

$$\Delta_n(f) = a^{2n-2}\prod_{i\neq j}(\alpha_i-\alpha_j)^2,$$

and $\{\alpha_i\}_{i=1}^n$ *are the roots of* f*, for* $n > 1$*, and* $\Delta_1(f) = 1$*, for any linear polynomial* f*. The number* $\Delta_n(f)$ *is called the discriminant of the polynomial. This relationship between different discriminants will play a major role in solving the cubic equation completely in section 3.*

A quick computation shows that if $f(\alpha) = 0$, then

$$\frac{f(x)}{x-\alpha} = \frac{f(x)-f(\alpha)}{x-\alpha} = ax^2 + (a\alpha+b)x + a\alpha^2 + b\alpha + c,$$

so that

$$\Delta_2(\frac{f}{x-\alpha}) = (a\alpha+b)^2 - 4a(a\alpha^2+b\alpha+c) = b^2-4ac-2ab\alpha-3a^2\alpha^2.$$

therefore, using the the notation of Theorem 2 we have

$$\Delta_2(\frac{f}{x-\alpha}) = b^2-4ac-2ab\alpha-3a^2\alpha^2, \tag{9}$$

Lemma 3. *Let* $f(x) = ax^3+bx^2+cx+d$ *such that* $b^2-3ac > 0$*. If* x_1 *and* x_2 *are the critical numbers of* f*, then*

$$f(x_1)f(x_2) = -\frac{\Delta_3(f)}{27a^2}. \tag{10}$$

and

$$f(x_1)+f(x_2) = \frac{-18abc+4b^3+54a^2d}{27a^2} = 2f(-\frac{b}{3a}) \tag{11}$$

Proof. If α, β, γ are the roots of f, then

$$f(x) = a(x-\alpha)(x-\beta)(x-\gamma),$$

from where

$$f'(x) = a(x-\beta)(x-\gamma) + a(x-\alpha)(x-\gamma) + (x-\alpha)(x-\beta),$$

from where it follows that

$$\begin{aligned} f'(\alpha) &= a(\alpha-\beta)(\alpha-\gamma), \\ f'(\beta) &= a(\beta-\alpha)(\beta-\gamma), \\ f'(\gamma) &= a(\gamma-\alpha)(\gamma-\beta), \end{aligned}$$

therefore

$$f'(\alpha)f'(\beta)f'(\gamma) = -a^3(\alpha-\beta)^2(\beta-\gamma)^2(\alpha-\gamma)^2 = -\frac{\Delta_3(f)}{a}.$$

On the other hand,

$$f'(x) = 3a(x - x_1)(x - x_2).$$

It follows that

$$\begin{aligned} f(x_1)f(x_2) &= a^2(x_1-\alpha)(x_1-\beta)(x_1-\gamma)(x_2-\alpha)(x_2-\beta)(x_2-\gamma) \\ &= a^2(x_1-\alpha)(x_2-\alpha)(x_1-\beta)(x_2-\beta)(x_1-\gamma)(x_2-\gamma) \\ &= a^2(x_1-\alpha)(x_2-\alpha)(x_1-\beta)(x_2-\beta)(x_1-\gamma)(x_2-\gamma) \\ &= a^2\tfrac{f'(\alpha)f'(\beta)f'(\gamma)}{27a^3} \\ &= -\tfrac{\Delta_3(f)}{27a^2}. \end{aligned}$$

On the other hand, by polynomial division we deduce that

$$9af(x) = (3ax+b)f'(x) - 2(b^2-3ac)x + 9ad - bc, \tag{12}$$

therefore

$$\begin{aligned} 9a(f(x_1)+f(x_2)) &= -2(b^2-3ac)(x_1+x_2) + 2(9ad-bc) \\ &= -2(b^2-3ac)\left(-\tfrac{2b}{3a}\right) + 2(9ad-bc) \\ &= \tfrac{-12abc+4b^3+54a^2d-6abc}{3a} \\ &= \tfrac{-12abc+4b^3+54a^2d-6abc}{3a} \\ &= \tfrac{-18abc+4b^3+54a^2d}{3a}, \end{aligned}$$

hence

$$f(x_1)+f(x_2) = \frac{-18abc+4b^3+54a^2d}{27a^2} = 2f(-\frac{b}{3a}),$$

where the last equality follows by equation (3). □

3. Roots of a Cubic Function

In this section we will study how to solve a cubic equation. Before we go into the analysis of the more general cases, let us assume that $f(x) = ax^3 + bx^2 + cx + d$ is a cubic function and that $b^2 - 3ac = 0$ or that $\Delta_3(f) = 0$, and let us solve the cubic equation in those cases.

In the case that $b^2 - 3ac = 0$, then

$$\begin{aligned} f(x) &= ax^3+bx^2+cx+d \\ &= ax^3+bx^2+\tfrac{b^2}{3a}x+d \\ &= a(x^3+\tfrac{b}{a}x^2+\tfrac{b^2}{3a^2}x)+d \\ &= a(x+\tfrac{b}{3a})^3 - \tfrac{b^3}{27a^2}+d, \end{aligned}$$

so that the real solution of the equation $f(x) = 0$ is

$$x = -\frac{b}{3a} + \sqrt[3]{\frac{b^3}{27a^3} - \frac{d}{a}} \tag{13}$$

In the case that $\Delta_3(f) = 0$, then there is a repeated root x that is a solution of the equation $f(x) = 0$ as well as a solution of the equation $f'(x) = 0$. Using equation (12) we conclude that if $f(x) = f'(x) = 0$, then

$$-2(b^2-3ac)x + 9ad - bc = 0.$$

If we assume that $b^2 - 3ac \neq 0$, then

$$x = \frac{9ad-bc}{2(b^2-3ac)}. \tag{14}$$

Since the sum of the all the solutions of the equation is $-b/a$, and this solution appears repeated at least twice, the third solution u satisfies

$$u + 2\left(\frac{9ad-bc}{2(b^2-3ac)}\right) = -\frac{b}{a},$$

so that

$$u = -\frac{b}{a} - \frac{9ad-bc}{b^2-3ac}$$

is the third solution.

In order to solve other cubic equations we separate any cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ into two parts. If $p(x) = ax^3 + bx^2 + cx$, then $f(x) = p(x) + d$. Observe that $f'(x) = p'(x)$ for all x.

In order to explain the method of solution, assume that x_0 is a solution of the equation $f(x) = 0$, that is $p(x_0) + d = 0$. If x_0 is not a critical number of p, then by the Implicit Function Theorem there exists a function $x = x(t)$ such that $p(x(t)) + t = 0$, and $x(d) = x_0$. Moreover, $x(t)$ is differentiable near $t = d$ and

$$x'(t) = -\frac{1}{p'(x(t))}. \tag{15}$$

What we will do is to find a new differential equation for $x(t)$ that can be solved near $t = d$. We will find solutions for the equation $p(x(t)) + t = 0$ in the zones where p (and hence f) is monotonic.

Assume that $x = x(t)$ is a solution of the equation $p(x(t)) + t = 0$, defined so that $x(t)$ is in the region where f (or p) is monotonic, then by equation (8) we have

$$\Delta_3(f) = f'(x(t))^2\Delta_2(\frac{f}{x - x(t)}),$$

so that

$$\Delta_3(f) = p'(x(t))^2\Delta_2(\frac{f}{x - x(t)}),$$

which implies by equation (15) that

$$\Delta_3(f)x'(t)^2 = \Delta_2(\frac{f}{x - x(t)}),$$

so that by equations (1) and (9) we conclude that

$$(c^2(b^2 - 4ac) + 2b(9ac - 2b^2)t - 27a^2t^2)x'(t)^2 = b^2 - 4ac - 2abx - 3a^2x^2. \tag{16}$$

Solving this equation is delicate, and we will show how to do that in the following pages.

Theorem 4. *Assume that a, b, c and d are real numbers such that $a > 0$ and $b^2 - 3ac < 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ has only one real solutions given by*

$$x = -\frac{b}{3a} - \frac{2\sqrt{3ac - b^2}}{3a}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d + 2b^3 - 9abc}{2(3ac - b^2)^{3/2}}\right)\right) \tag{17}$$

Proof. Let $p(x) = ax^3 + bx^2 + cx$, and let $f(x) = p(x) - p(-\frac{b}{3a})$, then $f(-\frac{b}{3a}) = 0$. Since $b^2 - 3ac < 0$ and $a > 0$, then $f'(x) > 0$ for all x.

Now in order to solve the equation $p(x) + d = 0$ we create the function $\alpha(t)$ that solves $p(\alpha(t)) + t = 0$, such that $\alpha(-p(-\frac{b}{3a})) = -\frac{b}{3a}$ We know, by the implicit function theorem that such function exists, it is differentiable and $\alpha'(-p(-\frac{b}{3a})) = -1/p'(\alpha) = -1/f'(\alpha) < 0$. It is not possible to have $\alpha'(t) = 0$ for any t because $p'(\alpha)\alpha'(t) + 1 = 0$, so that $\alpha'(t) < 0$ for any t.

First we multiply equation (16) by -1

$$(-c^2(b^2 - 4ac) - 2b(9ac - 2b^2)t + 27a^2t^2)x'(t)^2 = -b^2 + 4ac + 2abx + 3a^2x^2.$$

then we take square root and recall that $x'(t) < 0$ we get

$$-x'(t)\sqrt{-c^2(b^2 - 4ac) - 2b(9ac - 2b^2)t + 27a^2t^2} = \sqrt{-b^2 + 4ac + 2abx + 3a^2x^2},$$

from where it follows that the solution $\alpha(d)$ of this equation must satisfy

$$\int_{-\frac{b}{3a}}^{x(d)} \frac{du}{\sqrt{-b^2 + 4ac + 2abu + 3a^2u^2}} = -\int_{-p(-\frac{b}{3a})}^{d} \frac{dt}{\sqrt{-c^2(b^2 - 4ac) - 2b(9ac - 2b^2)t + 27a^2t^2}}. \tag{18}$$

Each of these integrals is computed by completing the square and performing a hyperbolic function substitution. Here are the highlights of these computations.

$$-b^2 + 4ac + 2abu + 3a^2u^2 = \frac{4}{3}(3ac - b^2) + 3a^2\left(u + \frac{b}{3a}\right)^2,$$

(19) $$a\sqrt{3}\left(u+\frac{b}{3a}\right)=\frac{2}{\sqrt{3}}\sqrt{3ac-b^2}\sinh\xi,$$

and
$$du=\frac{2}{3a}\sqrt{3ac-b^2}\cosh\xi\,d\xi.$$

The new limits of integration are $\xi=0$ when $u=-\frac{b}{3a}$, and
$$\xi=\sinh^{-1}\left(\frac{3a\alpha+b}{2\sqrt{3ac-b^2}}\right)$$
when $x=\alpha$. Then
$$\int_{-\frac{b}{3a}}^{\alpha}\frac{dx}{\sqrt{-b^2+4ac+2abx+3a^2x^2}}=\int_0^{\sinh^{-1}\left(\frac{3a\alpha+b}{2\sqrt{3ac-b^2}}\right)}\frac{1}{\frac{2}{\sqrt{3}}\sqrt{3ac-b^2}\cosh\xi}\frac{2}{3a}\sqrt{3ac-b^2}\cosh\xi\,d\xi.$$

Therefore

(20) $$\int_{-\frac{b}{3a}}^{\alpha}\frac{dx}{\sqrt{-b^2+4ac+2abx+3a^2x^2}}=\frac{1}{a\sqrt{3}}\sinh^{-1}\left(\frac{3a\alpha+b}{2\sqrt{3ac-b^2}}\right).$$

Now for the right hand side of (18) we complete the square first and write
$$-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2=-\frac{b^2(2b^2-9ac)^2+27a^2c^2(b^2-4ac)}{27a^2}+27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2.$$

Using identity (5) this simplifies to
$$-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2=4(3ac-b^2)^3+27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2.$$

Also
$$a\sqrt{27}\left(t+\frac{2b^3-9abc}{27a^2}\right)=\frac{2(3ac-b^2)^{3/2}}{a\sqrt{27}}\sinh\xi,$$

and
$$dt=\frac{2(3ac-b^2)^{3/2}}{27a^2}\cosh\xi\,d\xi.$$

The new limits of integration are
$$\xi=\sinh^{-1}\left(\frac{-27a^2p(-\frac{b}{3a})+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)=\sinh^{-1}0=0,$$
when $t=0$, and
$$\xi=\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)$$
when $t=d$. Therefore,
$$\int_{-p(-\frac{b}{3a})}^{d}\frac{dt}{\sqrt{-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2}}=\int_0^{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}\frac{\frac{2(3ac-b^2)^{3/2}}{27a^2}\cosh\xi}{\frac{2(3ac-b^2)^{3/2}}{a\sqrt{27}}\cosh\xi}\,d\xi.$$

It follows that,

(21) $$\int_{-p(-\frac{b}{3a})}^{d}\frac{dt}{\sqrt{-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2}}=\frac{1}{a\sqrt{27}}\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right).$$

Using equations (20) and (21) we can write the following equality instead
$$\frac{1}{a\sqrt{3}}\sinh^{-1}\left(\frac{3ax(d)+b}{2\sqrt{3ac-b^2}}\right)=-\frac{1}{a\sqrt{27}}\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)$$
from where equality (17) follows. □

Theorem 5. *Let a, b, c and d are real numbers such that $a > 0$ and let $f(x) = ax^3 + bx^2 + cx + d$. If $\Delta_3(f) > 0$, then the equation $f(x) = 0$ has three real solutions given by*

$$x_k = -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a} \sin\left(\frac{k\pi}{3} + \frac{(-1)^k}{3} \arcsin\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right) \tag{22}$$

where $k \in \{-1, 0, 1\}$, and $x_{-1} < x_0 < x_1$.

Proof. Recall that by equation (12)

$$9af(x) = (3ax + b)f'(x) - 2(b^2 - 3ac)x + 9ad - bc,$$

so that

$$9af(-\frac{b}{3a}) = -2(b^2 - 3ac)\left(-\frac{b}{3a}\right) + 9ad - bc.$$

Subtracting these equations we obtain

$$9a(f(x) - f(-\frac{b}{3a})) = (3ax + b)p'(x) - 2(b^2 - 3ac)\left(x + \frac{b}{3a}\right).$$

After multiplying by $3a$ we obtain

$$27a^2(f(x) - f(-\frac{b}{3a})) = (3ax + b)(3af'(x) - 2(b^2 - 3ac)) = (3ax + b)(9a^2x^2 + 6abx + 9ac - 2b^2)$$

We can factor the quadratic factor by solving the equation $9a^2x^2 + 6abx + 9ac - 2b^2 = 0$, whose solutions are

$$x = \frac{-6ab \pm \sqrt{36a^2b^2 - 36a^2(9ac - 2b^2)}}{18a^2} = \frac{-b \pm \sqrt{3(b^2 - 3ac)}}{3a}.$$

This means that the equation $f(x) = f(-\frac{b}{3a})$ has three solutions

$$\alpha = \frac{-b - \sqrt{3(b^2 - 3ac)}}{3a}, \beta = -\frac{b}{3a}, \gamma = \frac{-b + \sqrt{3(b^2 - 3ac)}}{3a}. \tag{23}$$

Observe that $b^2 - 3ac > 0$ since $\Delta_3(f) > 0$, so that the solutions are real numbers. Since $a > 0$, then $\alpha < \beta < \gamma$, and therefore $f'(\alpha) > 0$, $f'(\beta) < 0$, and $f'(\gamma) > 0$.

Now in order to solve the equation $p(x) + d = 0$ we create three functions, $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ solutions of the equations $p(x)+t = 0$ and such that such that $\alpha(-p(-\frac{b}{3a})) = \frac{-b-\sqrt{3(b^2-3ac)}}{3a}$, $\beta(-p(-\frac{b}{3a})) = -\frac{b}{3a}$, and $\gamma(-p(-\frac{b}{3a})) = \frac{-b+\sqrt{3(b^2-3ac)}}{3a}$. We know, by the implicit function theorem, that such functions exist, are differentiable and $\alpha'(-p(-\frac{b}{3a})) = -1/p'(\alpha) = -1/f'(\alpha) < 0$, $\beta'(-p(-\frac{b}{3a})) = -1/p'(\beta) = -1/f'(\beta) > 0$, and $\gamma'(-p(-\frac{b}{3a})) = -1/p'(\gamma) = -1/f'(\gamma) < 0$. Given that it is not possible to have $x'(t) = 0$ for any t (because $p'(x)x'(t) + 1 = 0$) it follows that $\alpha'(t) < 0$, $\beta'(t) > 0$, and $\gamma'(t) < 0$ for all t (again, by the Implicit Function Theorem, α, β, and γ have second derivatives, so their first derivatives are continuous and do not change signs.)

In order to avoid repeating the same computation three times we will do it in general one time. Let us return to equation (16). We solve the equation in the region where Δ (seen as a function of t) is positive. As we have seen $x'(t)$ has one, and only one sign, but it could be either positive or negative. This means the we can solve the separable differential equation (16) through integrating

$$\int_{x(-p(-\frac{b}{3a}))}^{x(d)} \frac{du}{\sqrt{b^2 - 4ac - 2abu - 3a^2u^2}} = \pm \int_{-p(-\frac{b}{3a})}^{d} \frac{dt}{\sqrt{c^2(b^2 - 4ac) + 2b(9ac - 2b^2)t - 27a^2t^2}}, \tag{24}$$

where the choice of sign is the same as the choice of sign for $x'(t)$.

Each of these integrals is computed by completing the square and performing a trigonometric substitution. Here are the highlights of these computations.

$$b^2 - 4ac - 2abu - 3a^2u^2 = \frac{4}{3}(b^2 - 3ac) - 3a^2\left(u + \frac{b}{3a}\right)^2,$$

$$a\sqrt{3}\left(u+\frac{b}{3a}\right)=\frac{2}{\sqrt{3}}\sqrt{b^2-3ac}\sin\theta. \tag{25}$$

and

$$du=\frac{2}{3a}\sqrt{b^2-3ac}\cos\theta\,d\theta.$$

The new limits of integration are

$$\theta=\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)$$

when $u=x(-p(-\frac{b}{3a}))$, and

$$\theta=\arcsin\left(\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right)$$

when $u=x(d)$. Then

$$\int_{x(-p(-\frac{b}{3a}))}^{x(d)}\frac{du}{\sqrt{b^2-4ac-2abu-3a^2u^2}}=\int_{\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)}^{\arcsin\left(\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right)}\frac{\frac{2}{3a}\sqrt{b^2-3ac}\cos\theta}{\frac{2}{\sqrt{3}}\sqrt{b^2-3ac}\cos\theta}\,d\theta.$$

Therefore

$$\int_{x(-p(-\frac{b}{3a}))}^{x(d)}\frac{du}{\sqrt{b^2-4ac-2abu-3a^2u^2}}=\frac{\arcsin\left(\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right)-\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)}{a\sqrt{3}}. \tag{26}$$

Now for the right hand side of (24) we complete the square first and write

$$c^2(b^2-4ac)+2b(9ac-2b^2)t-27a^2t^2=\frac{b^2(2b^2-9ac)^2+27a^2c^2(b^2-4ac)}{27a^2}-27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2.$$

Using identity (5) this simplifies to

$$c^2(b^2-4ac)+2b(9ac-2b^2)t-27a^2t^2=4(b^2-3ac)^3-27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2.$$

Also

$$a\sqrt{27}\left(t+\frac{2b^3-9abc}{27a^2}\right)=\frac{2(b^2-3ac)^{3/2}}{a\sqrt{27}}\sin\theta,$$

and

$$dt=\frac{2(b^2-3ac)^{3/2}}{27a^2}\cos\theta\,d\theta.$$

The new limits of integration are

$$\theta=\arcsin\left(\frac{-27a^2p(-\frac{b}{3a})+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)=0$$

when $t=-p(-\frac{b}{3a})$, and

$$\theta=\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)$$

when $t=d$. Therefore,

$$\int_{-p(-\frac{b}{3a})}^{d}\frac{dt}{\sqrt{c^2(b^2-4ac)+2b(9ac-2b^2)t-27a^2t^2}}=\int_0^{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}\frac{\frac{2(b^2-3ac)^{3/2}}{27a^2}\cos\theta}{\frac{2(b^2-3ac)^{3/2}}{a\sqrt{27}}\cos\theta}\,d\theta.$$

Therefore,

$$\int_{-p(-\frac{b}{3a})}^{d}\frac{dt}{\sqrt{c^2(b^2-4ac)+2b(9ac-2b^2)t-27a^2t^2}}=\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{a\sqrt{27}}. \tag{27}$$

Substituting equations (26) and (27) into (24) yields

$$\frac{\arcsin\left(\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right)-\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)}{a\sqrt{3}}=\pm\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{a\sqrt{27}}. \tag{28}$$

from where it follows that

$$x(d)=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\sin\left(\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)\pm\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right).$$

Now we make the choices of solutions. When we choose $x(d)=\alpha(d)$, then we need to pick the negative sign and

$$\begin{aligned}\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right) &= \arcsin\left(\frac{3a\alpha(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)=\arcsin\left(\frac{-\sqrt{3(b^2-3ac)}}{2\sqrt{b^2-3ac}}\right)\\ &= \arcsin\left(\frac{-\sqrt{3}}{2}\right)=-\frac{\pi}{3},\end{aligned}$$

therefore

$$\alpha(d)=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\sin\left(-\frac{\pi}{3}-\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right). \tag{29}$$

When we choose $x(d)=\beta(d)$, then we need to pick the positive sign and

$$\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)=\arcsin\left(\frac{3a\beta(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)=\arcsin\left(\frac{0}{2\sqrt{b^2-3ac}}\right)=0,$$

therefore

$$\beta(d)=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\sin\left(\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right). \tag{30}$$

Finally, when we choose $x(d)=\gamma(d)$, then we need to pick the positive sign and

$$\begin{aligned}\arcsin\left(\frac{3ax(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right) &= \arcsin\left(\frac{3a\gamma(-p(-\frac{b}{3a}))+b}{2\sqrt{b^2-3ac}}\right)=\arcsin\left(\frac{\sqrt{3(b^2-3ac)}}{2\sqrt{b^2-3ac}}\right)\\ &= \arcsin\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3},\end{aligned}$$

therefore

$$\gamma(d)=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\sin\left(\frac{\pi}{3}-\frac{\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right). \tag{31}$$

Formulas (29), (30), and (31) can be combined into one formula as in equation (22). The fact that $x_{-1}<x_0<x_1$ follows from the deduction of the formulas, since x_0 is between the critical points of $f'(x)$, while x_{-1} is smaller than the critical points of f and x_1 is bigger than the critical points of f, according to the deduction of α, β, and γ. □

The following corollary shows that solutions of a cubic equation with positive discriminant can be found in an interval of length $\frac{4\sqrt{b^2-3ac}}{3a}$ centered at $-\frac{b}{3a}$. This is twice the length of the interval whose end points are the critical points of the corresponding polynomial function.

Corollary 5.1. *Let a, b, c, and d be real numbers such that $a > 0$, and let $f(x) = ax^3 + bx^2 + cx + d$. Assume additionally that $\Delta_3(f) > 0$. If x is a solution of the equation $f(x) = 0$, then*

$$-\frac{b}{3a} - \frac{2\sqrt{b^2 - 3ac}}{3a} \le x \le -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a}. \tag{32}$$

Proof. This follows from (22) by noticing that $|\sin x| \le 1$ for any x. ☐

Another way to state the previous Corollary is to say that if a cubic equation with real coefficients, $ax^3 + bx^2 + cx + d = 0$ has a real solution x such that $|x + \frac{b}{3a}| > \frac{2\sqrt{b^2-3ac}}{3|a|}$, then the other two solutions must be complex conjugate solutions.

In order to shorten our notation we will denote by x_3 and x_4 the left and right bounds above, respectively. See Section 5 to see the location of these points with respect to the graph of the function f.

Remark 5.1. *The bounds for the solutions of a cubic equation with positive discriminant are the solutions of the quadratic equation*

$$(3ax + b)^2 - 4(b^2 - 3ac) = 0.$$

This implies that if $x < x_3$ or $x > x_4$, then $(3ax + b)^2 - 4(b^2 - 3ac) > 0$.

Remark 5.2. *In the case that $b^2 - 3ac > 0$, it follows that $f(x_1) = f(x_4)$ and $f(x_2) = f(x_3)$. In order to prove this note that x_2 is a double solution of the equation $f(x) = f(x_2)$, since f has a critical number at x_2. The third solultion x satisfies*

$$x_2 + x_2 + x = -\frac{b}{a},$$

from where

$$x = -\frac{b}{a} - 2x_2 = -\frac{b}{a} - 2\left(\frac{-b + \sqrt{b^2 - 3ac}}{3a}\right) = \frac{-b - 2\sqrt{b^2 - 3ac}}{3a} = x_3.$$

A similar argument can be used to show that $f(x_1) = f(x_4)$.

The last case we have to consider is when $\Delta_3(f) < 0$ but $b^2 - 3ac > 0$.

Theorem 6. *Let a, b, c and d be real numbers. Let $f(x) = ax^3 + bx^2 + cx + d$ and let Δ be the discriminant of the cubic equation $f(x) = 0$. Assume that $a > 0$, $\Delta_3(f) < 0$, and $b^2 - 3ac > 0$, then there is only one real solution of the equation $f(x) = 0$, given by*

$$x = -\frac{b}{3a} - \frac{2\sqrt{b^2 - 3ac}}{3a} \cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right) \tag{33}$$

when $27a^2d + 2b^3 - 9abc > 0$, and

$$x = -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a} \cosh\left(\frac{1}{3}\cosh^{-1}\left(-\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right) \tag{34}$$

when $27a^2d + 2b^3 - 9abc < 0$.

Proof. Since $b^2 - 3ac > 0$, there is a local maximum at some number x_1 and a local minimum at some number x_2. Since $a > 0$, then $x_1 < x_2$.

First we establish that if x_0 is a solution of the equation $f(x) = 0$, then either $x_0 < x_1$ or $x_0 > x_2$. We cannot have $x_0 = x_1$ or $x_0 = x_2$, because this would make x_0 a double root of f, which would imply that $\Delta_3(f) = 0$, which is a contradiction. Similarly we cannot have $x_1 < x_0 < x_2$ because since f is decreasing in the interval $[x_1, x_2]$, then $f(x_1) > f(x_0) = 0 > f(x_2)$. Since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, then by the Intermediate Value Theorem there exist $u < x_1$ and $v > x_2$ such that $f(u) = f(v) = 0$. This implies that $\Delta_3(f) > 0$, which is a contradiction. Therefore, if x_0 solves $f(x) = 0$, then either $x_0 < x_1$ or $x_0 > x_2$. Therefore $f'(x_0) > 0$.

Let $p(x) = ax^3 + bx^2 + cx$. If $x(t)$ is the solution of $p(x(t)) + t = 0$, then $x'(t) = -1/p'(x(t)) = -1/f'(x(t)) < 0$.

If $x_0 < x_3$ or $x_0 > x_4$, then by equation (16)

$$\int_{x_0}^{x(d)} \frac{du}{\sqrt{-b^2+4ac+2abu+3a^2u^2}} = -\int_{-p(x_0)}^{d} \frac{dt}{\sqrt{-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2}}, \tag{35}$$

Each of these integrals is computed by completing the square and performing a hyperbolic function substitution. Here are the highlights of these computations.

$$-b^2+4ac+2abu+3a^2u^2 = 3a^2\left(u+\frac{b}{3a}\right)^2 - \frac{4}{3}(b^2-3ac)$$

$$a\sqrt{3}\left|u+\frac{b}{3a}\right| = \frac{2}{\sqrt{3}}\sqrt{b^2-3ac}\cosh\xi,$$

and

$$du = \sigma\frac{2}{3a}\sqrt{b^2-3ac}\sinh\xi\, d\xi,$$

where $\sigma = \operatorname{sign}(x_0+\frac{b}{3a})$. Observe that the sign of $u+\frac{b}{3a}$ is constant in the intervals $(-\infty, x_3)$ and (x_4,∞), so we use $\sigma = \operatorname{sign}(x_0+\frac{b}{3a})$. The new limits of integration are

$$\xi = \cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right|$$

when $x = x(d)$, and

$$\xi = \cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right|$$

when $x = x_0$. Then

$$\int_{x_0}^{x(d)} \frac{dx}{\sqrt{-b^2+4ac+2abx+3a^2x^2}} = \int_{\cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right|}^{\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right|} \frac{1}{\frac{2\sigma}{\sqrt{3}}\sqrt{b^2-3ac}\sinh\xi}\frac{2}{3a}\sqrt{b^2-3ac}\sinh\xi\, d\xi.$$

Therefore

$$\int_{x_0}^{x(d)} \frac{dx}{\sqrt{-b^2+4ac+2abx+3a^2x^2}} = \frac{\sigma}{a\sqrt{3}}\left(\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| - \cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right|\right) \tag{36}$$

On the other hand, the computation of the right hand side of (35) is done in a similar way. First we complete the square and write

$$-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2 = 27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2 - \frac{b^2(2b^2-9ac)^2+27a^2c^2(b^2-4ac)}{27a^2}$$

but by idenity (5)

$$b^2(2b^2-9ac)^2+27a^2c^2(b^2-4ac) = 4(b^2-3ac)^3,$$

so that

$$-c^2(b^2-4ac)-2b(9ac-2b^2)t+27a^2t^2 = 27a^2\left(t+\frac{2b^3-9abc}{27a^2}\right)^2 - 4(b^2-3ac)^3.$$

We also make the substitution

$$a\sqrt{27}\left|t+\frac{2b^3-9abc}{27a^2}\right| = \frac{2(b^2-3ac)^{3/2}}{a\sqrt{27}}\cosh\xi,$$

and

$$dt = \mu\frac{2(b^2-3ac)^{3/2}}{27a^2}\sinh\xi\, d\xi,$$

where $\mu = \operatorname{sign}(p(x_0)+\frac{2b^3-9abc}{27a^2})$. Here the argument is more delicate, but it goes along the following lines.

$$p(x)+\frac{2b^3-9abc}{27a^2} = f(x)-f(-\frac{b}{3a}),$$

which has constant sign whenever $x < x_3$ or $x > x_4$, so it is equal to the sign of $p(x_0) + \frac{2b^3-9abc}{27a^2}$. In fact, if $x < x_3$, then $f(x) < f(x_3) < f(-\frac{b}{3a})$, and if $x > x_4$, then $f(x) > f(x_4) > f(-\frac{b}{3a})$.

The new limits of integration are

$$\xi = \cosh^{-1}\left|\frac{-27a^2p(x_0) + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right|$$

when $t = -p(x_0)$, and

$$\xi = \cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right|$$

when $t = d$. Therefore,

$$\int_{-p(x_0)}^{d} \frac{dt}{\sqrt{c^2(b^2-4ac) + 2b(9ac-2b^2)t - 27a^2t^2}} = \int_{\cosh^{-1}\left|\frac{-27a^2p(x_0)+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|}^{\cosh^{-1}\left|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|} \frac{\mu\frac{2(b^2-3ac)^{3/2}}{27a^2}\sinh\xi}{\frac{2(b^2-3ac)^{3/2}}{a\sqrt{27}}\sinh\xi}\, d\xi.$$

Therefore,

$$\int_{-p(x_0)}^{d} \frac{dt}{\sqrt{-c^2(b^2-4ac) - 2b(9ac-2b^2)t + 27a^2t^2}} = \frac{\mu}{a\sqrt{27}}\,\xi\Big|_{\cosh^{-1}\left|\frac{-27a^2p(x_0)+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|}^{\cosh^{-1}\left|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|} \tag{37}$$

This is equal to

$$= \frac{\mu}{a\sqrt{27}}\left(\cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| - \cosh^{-1}\left|\frac{-27a^2p(x_0) + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right|\right) \tag{38}$$

Observe that if $x_0 < x_3$, then $\sigma = \mu = -1$, and if $x_0 > x_4$, then $\sigma = \mu = 1$, so $\sigma = \mu$ in any case.

Now we are ready to start solving the cubic equation. First, by equations (35), (36), and (38) we obtain

$$\begin{aligned} &\frac{\sigma}{a\sqrt{3}}\left(\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| - \cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right|\right) \\ = &\frac{\mu}{a\sqrt{27}}\left(\cosh^{-1}\left|\frac{-27a^2p(x_0) + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| - \cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right|\right) \end{aligned} \tag{39}$$

This equation can be rewritten more simply as

$$\begin{aligned} &\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| - \cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right| \\ = &\frac{1}{3}\left(\cosh^{-1}\left|\frac{-27a^2p(x_0) + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| - \cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right|\right) \end{aligned} \tag{40}$$

Let us rearrange the terms in the previous equation so that the terms that contain x_0 are in the right hand side of the equation, and those that do not are in the left hand side of the equation. We obtain

$$\begin{aligned} &\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| + \frac{1}{3}\cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| \\ = &\cosh^{-1}\left|\frac{3ax_0+b}{2\sqrt{b^2-3ac}}\right| + \frac{1}{3}\cosh^{-1}\left|\frac{-27a^2p(x_0) + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| \end{aligned} \tag{41}$$

Equation (41) shows that the right hand side of (41) is constant and does not depend on x_0. In fact, we will prove that it is equal to zero.

In order to do that we first notice that

$$\begin{aligned} -27a^2p(x) + 2b^3 - 9abc &= -(3ax+b)(9a^2x^2 + 6abx + 9ac - 2b^2) \\ &= -(3ax+b)((3ax+b)^2 - 3(b^2-3ac)) \\ &= -(3ax+b)((3ax+b)^2 - 4(b^2-3ac) + (b^2-3ac)) \end{aligned} \tag{42}$$

Notice now that if $x = x_3$ or $x = x_4$, then $(3ax+b)^2 - 4(b^2-3ac) = 0$, so that

$$-27a^2p(x_3) + 2b^3 - 9abc = -(3ax_3+b)(b^2-3ac) = 2(b^2-3ac)^{3/2}, \tag{43}$$

and

$$-27a^2p(x_4)+2b^3-9abc=-(3ax_4+b)(b^2-3ac)=-2(b^2-3ac)^{3/2}. \tag{44}$$

Now we are ready to prove that the right hand side of equation (41) is equal to 0. The plan is to prove that both the limit as x_0 approaches x_3 from the left and the limit as x_0 approaches x_4 from the right of the right hand side of equation (41) is 0. Both limits can be computed by evaluation.

Let us handle each case separately.

$$\begin{aligned}
&\lim_{x\to x_3^-}\left(\cosh^{-1}|\frac{3ax+b}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{-27a^2p(x)+2b^3-9abc}{2(b^2-3ac)^{3/2}}|\right)\\
=\;&\cosh^{-1}|\frac{3ax_3+b}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{-27a^2p(x_3)+2b^3-9abc}{2(b^2-3ac)^{3/2}}|\\
=\;&\cosh^{-1}|-\frac{2\sqrt{b^2-3ac}}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{2(b^2-3ac)^{3/2}}{2(b^2-3ac)^{3/2}}|\\
=\;&\cosh^{-1}(1)+\tfrac{1}{3}\cosh^{-1}(1)=0+\tfrac{1}{3}\cdot 0=0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{x\to x_4^+}\left(\cosh^{-1}|\frac{3ax+b}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{-27a^2p(x)+2b^3-9abc}{2(b^2-3ac)^{3/2}}|\right)\\
=\;&\cosh^{-1}|\frac{3ax_4+b}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{-27a^2p(x_4)+2b^3-9abc}{2(b^2-3ac)^{3/2}}|\\
=\;&\cosh^{-1}|\frac{2\sqrt{b^2-3ac}}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|-\frac{2(b^2-3ac)^{3/2}}{2(b^2-3ac)^{3/2}}|\\
=\;&\cosh^{-1}(1)+\tfrac{1}{3}\cosh^{-1}(1)=0+\tfrac{1}{3}\cdot 0=0.
\end{aligned}$$

This means that the solution satisfies the simpler equation

$$\cosh^{-1}|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}|+\frac{1}{3}\cosh^{-1}|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}|=0.$$

Therefore,

$$\cosh^{-1}\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right|=-\frac{1}{3}\cosh^{-1}\left|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right| \tag{45}$$

The key observation about the expressions above is that the sign of $27a^2d+2b^3-9abc$ is the opposite of the sign of $3ax(d)+b$. In order to prove this, notice that by equation (42)

$$27a^2d+2b^3-9abc=-(3ax(d)+b)((3ax(d)+b)^2-4(b^2-3ac)+(b^2-3ac))$$

and we can see that the second factor above is positive because it is the sum of $(3ax+b)^2-4(b^2-3ac)$ and b^2-3ac which is a sum of two positive numbers when $x(d)<x_3$ or $x(d)>x_4$ (see Remark 5.1).

In the case that $x(d)<x_3<-\frac{b}{3a}$, $3ax(d)+b<0$, while $27a^2p(x_3)+2b^3-9abc>0$, so equation (45) implies that the solution $x=x(d)$ satisfies

$$\cosh^{-1}\left(-\frac{3ax+b}{2\sqrt{b^2-3ac}}\right)=-\frac{1}{3}\cosh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)$$

From where equation (33) follows.

Notice that in this case, since $x(d)<x_3<x_1$, it follows that $f(x(d))<f(x_1)$, since f is monotonic in the interval $(-\infty,x_1]$, so that $f(x_1)>0$. Conversely, if $f(x_1)>0$, since $f(x_1)f(x_2)=-\frac{\Delta_3(f)}{27a^2}>0$, then $f(x_2)>0$ and since $f(x_3)=f(x_2)$ it follows that $f(x_3)>0$. Since f in increasing in the interval $[x_3,x_1]$, decreasing in the interval $[x_1,x_2]$, and increasing in the interval $[x_2,\infty)$, then the minimum value of f in the interval $[x_3,\infty)$ is $f(x_2)>0$. This means that if $f(x)=0$, then $x<x_3$, that is under the hypothesis that $\Delta_3(f)<0$ and $b^2-3ac>0$, the solution of the equation $f(x)=0$ satisfies $x<x_3$ if and only if $f(x_1)>0$. Finally, since both $f(x_1)$ and $f(x_2)$ have the same sign, they both have the sign of the right hand side of equation (11), which is the same sign as the sign of $-9abc+2b^3+27a^2d$. This means that under the conditions of the theorem, the solution of the equation $f(x) = 0$ satisfies $x < x_3$ if and only if $27a^2d + 2b^3 - 9abc > 0$.

In the case that $x(d) > x_4 > -\frac{b}{3a}$, $3ax(d) + b > 0$, while $27a^2p(x_3) + 2b^3 - 9abc < 0$, so equation (45) implies that the solution $x = x(d)$ satisfies

$$\cosh^{-1}\left(\frac{3ax+b}{2\sqrt{b^2-3ac}}\right) = -\frac{1}{3}\cosh^{-1}\left(-\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)$$

From where equation (34) follows.

Notice that in this case, since $x(d) > x_4 > x_2$, it follows that $f(x(d)) > f(x_2)$, since f is monotonic in the interval $[x_2, \infty)$, so that $f(x_2) < 0$. Since $f(x_1)f(x_2) = -\frac{\Delta_3(f)}{27a^2} > 0$, then $f(x_1) < 0$. Conversely, if $f(x_1) < 0$, then $f(x_4) = f(x_1) < 0$. Since f in increasing in the interval $(-\infty, x_1)$, decreasing in the interval $[x_1, x_2]$, and increasing in the interval $[x_2, x_4]$, then the maximum value of f in the interval $(-\infty, x_4]$ is $f(x_1) < 0$. This means that if $f(x) = 0$, then $x > x_4$, that is under the hypothesis that $\Delta_3(f) < 0$ and $b^2 - 3ac > 0$, the solution of the equation $f(x) = 0$ satisfies $x > x_4$ if and only if $f(x_1) < 0$. Finally, since both $f(x_1)$ and $f(x_2)$ have the same sign, they both have the sign of the right hand side of equation (11), which is the same sign as the sign of $-9abc + 2b^3 + 27a^2d$. This means that under the conditions of the theorem, the solution of the equation $f(x) = 0$ satisfies $x > x_4$ if and only if $27a^2d + 2b^3 - 9abc < 0$. ☐

A summary of the formulas for solving a cubic equation appear in Figure 2.

Example A famous example considered by Bombelli in the discovery of complex numbers is the equation $x^3 - 15x - 4 = 0$. This equation has 3 real roots. The biggest of these is the number 4. This means that in order to produce such solution we need to use formula (22) with $k = 1$. When doing so we compute $b^2 - 3ac = 45$ and $27a^2d + 2b^3 - 9abc = -108$. Applying the formula produces the identity

$$2\sqrt{5}\sin\left(\frac{\pi + \arcsin\frac{2\sqrt{5}}{25}}{3}\right) = 4.$$

Another way to write this identity is as

$$3\arcsin\left(\frac{2\sqrt{5}}{5}\right) - \arcsin\left(\frac{2\sqrt{5}}{25}\right) = \pi.$$

4. Radical Solutions of a Cubic Equation

When we solved the cubic equation, only one of the formulas we found was expressed in terms in radicals. However, some of the formulas we found in section 3 can be expressed in terms of radicals, albeit, in a more complex form.

In order to show how to do that, let us start by looking at the simplest case which is to use equation (45). From that equation it follows that

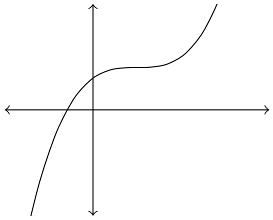
$$\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| = \cosh\left(\frac{1}{3}\cosh^{-1}\left|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|\right)$$

Now observe that for any number $p \geq 1$ we have

$$\begin{aligned}
\cosh\left(\tfrac{1}{3}\cosh^{-1}(|p|)\right) &= \tfrac{1}{2}\left(e^{\frac{\cosh^{-1}(|p|)}{3}} + e^{-\frac{\cosh^{-1}(|p|)}{3}}\right) \\
&= \tfrac{1}{2}\left(e^{\frac{\ln(\sqrt{p^2-1}+|p|)}{3}} + e^{-\frac{\ln(\sqrt{p^2-1}+|p|)}{3}}\right) \\
&= \tfrac{1}{2}\left(\sqrt[3]{\sqrt{p^2-1}+|p|} + \frac{1}{\sqrt[3]{\sqrt{p^2-1}+|p|}}\right).
\end{aligned}$$

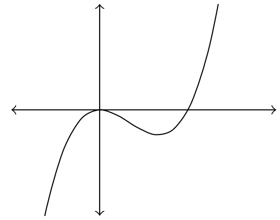
In the case of the cubic equation

$$p = \frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}},$$



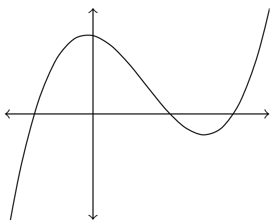
$b^2-3ac=0,$

$x=-\frac{b}{3a}+\sqrt[3]{\frac{b^3}{27a^3}-\frac{d}{a}}$



$\Delta_3(f)=0,\ b^2-3ac\neq 0,$

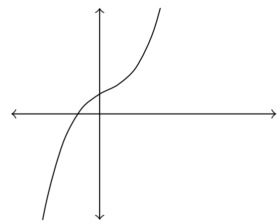
$x=\frac{9ad-bc}{2(b^2-3ac)},$
$x=-\frac{b}{a}-\frac{9ad-bc}{b^2-3ac}$



$a>0,\ \Delta_3(f)>0,$

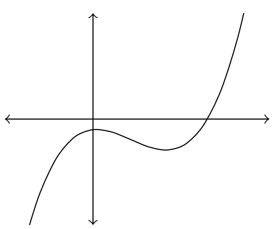
$$x_k=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\sin\left(\frac{k\pi+(-1)^k\arcsin\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right)$$

where $k\in\{-1,0,1\}$, and $x_{-1}<x_0<x_1$.



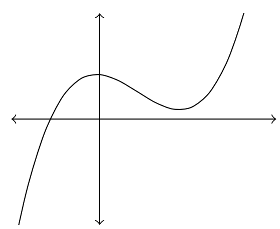
$b^2-3ac<0,$

$$x=-\frac{b}{3a}-\frac{2\sqrt{3ac-b^2}}{3a}\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)$$



$a>0,\ \Delta_3(f)<0,\ b^2-3ac>0,$
$27a^2d+2b^3-9abc<0,$

$$x=-\frac{b}{3a}+\frac{2\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}\left(-\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right)$$



$a>0,\ \Delta_3(f)<0,\ b^2-3ac>0,$
$27a^2d+2b^3-9abc>0$

$$x=-\frac{b}{3a}-\frac{2\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right)$$

FIGURE 2. Summary of formulas to find real solutions of the equation $ax^3+bx^2+cx+d=0$.

so that, using identity (2) we obtain

$$\begin{aligned} p^2-1 &= \frac{(27a^2d+2b^3-9abc)^2-4(b^2-3ac)^3}{4(b^2-3ac)^3}=\frac{-27a^2(b^2c^2-4ac^3-4b^3d+18abcd-27a^2d^2)}{4(b^2-3ac)^3}\\ &= \frac{-27a^2\Delta_3(f)}{4(b^2-3ac)^3}, \end{aligned}$$

therefore,

$$\sqrt[3]{\sqrt{p^2-1}+|p|} = \sqrt[3]{\sqrt{\frac{-27a^2\Delta_3(f)}{4(b^2-3ac)^3}} + \left|\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right|} = \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+|27a^2d+2b^3-9abc|}}{\sqrt[3]{2}\sqrt{b^2-3ac}}.$$

It follows that

$$\left|\frac{3ax(d)+b}{2\sqrt{b^2-3ac}}\right| = \frac{1}{2}\left[\frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+|27a^2d+2b^3-9abc|}}{\sqrt[3]{2}\sqrt{b^2-3ac}} + \frac{\sqrt[3]{2}\sqrt{b^2-3ac}}{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+|27a^2d+2b^3-9abc|}}\right].$$

Simplifying, we obtain

$$|3ax(d)+b| = \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+|27a^2d+2b^3-9abc|}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}(b^2-3ac)}{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+|27a^2d+2b^3-9abc|}}.$$

In the case the $x < x_3$, the previous equality yields the solution

$$x = -\frac{b}{3a} - \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(b^2-3ac)}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}. \tag{46}$$

Similarly, in the case the $x > x_4$, the previous equality yields the solution

$$x = -\frac{b}{3a} + \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}-27a^2d-2b^3+9abc}}{3a\sqrt[3]{2}} + \frac{\sqrt[3]{2}(b^2-3ac)}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}-27a^2d-2b^3+9abc}}. \tag{47}$$

Now let us write equation (17) in terms of radicals. In order to do so we use the identity

$$\sinh\left(\frac{\sinh^{-1}(p)}{3}\right) = \frac{1}{2}\left(\sqrt[3]{\sqrt{p^2+1}+p} - \frac{1}{\sqrt[3]{\sqrt{p^2+1}+p}}\right). \tag{48}$$

When we use this formula with

$$p = \frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}},$$

we obtain

$$\begin{aligned} p^2+1 &= \frac{(27a^2d+2b^3-9abc)^2+4(3ac-b^2)^3}{4(3ac-b^2)^3} = \frac{-27a^2(b^2c^2-4ac^3-4b^3d+18abcd-27a^2d^2)}{4(3ac-b^2)^3} \\ &= \frac{-27a^2\Delta_3(f)}{4(3ac-b^2)^3}, \end{aligned}$$

therefore,

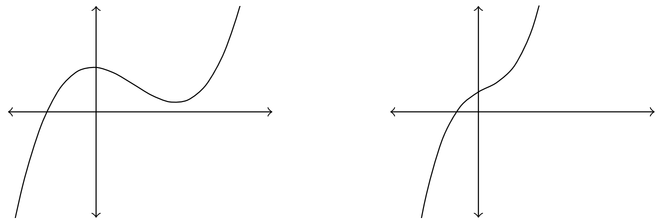
$$\sqrt[3]{\sqrt{p^2+1}+p} = \sqrt[3]{\sqrt{\frac{-27a^2\Delta_3(f)}{4(3ac-b^2)^3}} + \frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}} = \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}{\sqrt[3]{2}\sqrt{3ac-b^2}}. \tag{49}$$

It follows that equation (17) can be written in terms of radicals as

$$x = -\frac{b}{3a} - \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}{3a\sqrt[3]{2}} + \frac{\sqrt[3]{2}(3ac-b^2)}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}. \tag{50}$$

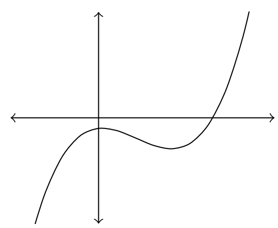
Observe that formulas (46) and (50) are the same formula.

Figure 3 shows a summary of the formulas we have found in this section using radicals.



$a > 0$, $\Delta_3(f) < 0$, $b^2 - 3ac > 0$, and $27a^2d + 2b^3 - 9abc > 0$; or $b^2 - 3ac < 0$.

$$x = -\frac{b}{3a} - \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(b^2 - 3ac)}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}$$



$a > 0$, $\Delta_3(f) < 0$, $b^2 - 3ac > 0$, and $27a^2d + 2b^3 - 9abc < 0$.

$$x = -\frac{b}{3a} + \frac{\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} - (27a^2d + 2b^3 - 9abc)}}{3a\sqrt[3]{2}} + \frac{\sqrt[3]{2}(b^2 - 3ac)}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} - (27a^2d + 2b^3 - 9abc)}}.$$

FIGURE 3. Formulas to solve the equation $ax^3 + bx^2 + cx + d = 0$ using radicals.

5. More on the Graph of a Cubic Function

Cubic functions have a symmetry with respect to its inflection point. Specifically this means that

$$f\left(-\frac{b}{3a} + h\right) + f\left(-\frac{b}{3a} - h\right) = 2f\left(-\frac{b}{3a}\right). \tag{51}$$

for any number h. This can be proved by noticing that the function

$$g(h) = f\left(-\frac{b}{3a} + h\right) + f\left(-\frac{b}{3a} - h\right)$$

is even, and

$$\begin{aligned} g(0) &= 2f\left(-\tfrac{b}{3a}\right), \\ g'(0) &= 0, \text{ since } f \text{ is even,} \\ g''(0) &= 2f''(-\tfrac{b}{3a}) = 0, \\ g'''(0) &= 0, \text{ since } f \text{ is even, and} \\ g^{(n)}(0) &= 0, \text{ for } n > 3, \text{ since } g \text{ has at most degree 3.} \end{aligned}$$

Therefore $g(h) = g(0)$ by Taylor expansion. In particular this explains why the cubic function has its local maximum and minimum symmetric with respect to the inflection point.

On the other hand, if $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers is a cubic function, during our work to solve the cubic equation $f(x) = 0$ in the case that $\Delta_3(f) > 0$ we denoted by x_1 and x_2 the critical points of f, and denoted by x_3 and x_4 the end points of an interval that contained all three solutions of the cubic equation. In the case that $a > 0$ these points are sorted as follows

$$x_3 < x_1 < -\frac{b}{3a} < x_2 < x_4.$$

The distance between each two consecutive points in this sequence is $\delta = \frac{\sqrt{b^2-3ac}}{3a}$. In addition, $f(x_1) = f(x_4)$, and $f(x_2) = f(x_3)$. If we look at the formulas we deduced to find the roots of f in this case, we notice that the solutions use circular functions, so we call this part of the graph the sinusoidal zone of the graph. However, in order to obtain the solutions of f outside of the sinusoidal zone of the graph we use hyperbolic functions, so we refer to this zone as the hyperbolic zone.

A closer look at the sinusoidal zone of a cubic function appears in Figure 4.

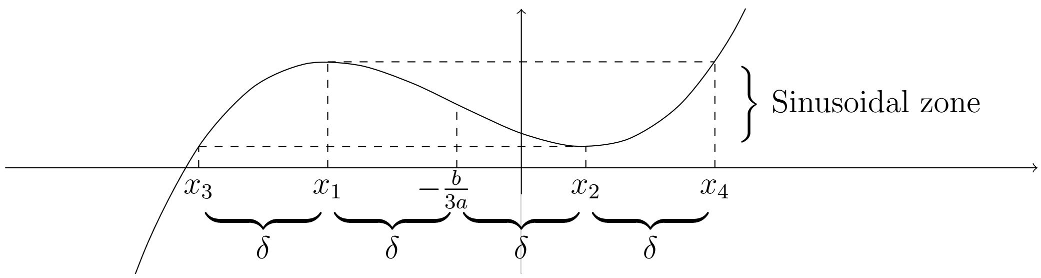


FIGURE 4. The sinusoidal zone of a cubic function.

A deeper look at the sinusoidal zone of a cubic function can be made if we parametrize this zone using the sine function, that is, if x is such that $x_3 \leq x \leq x_4$, then we can write

$$x = -\frac{b}{3a} + \frac{2\sqrt{b^2-3ac}}{3a}\sin\theta, \tag{52}$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This is the change of variable that we did in equation (25) to solve the differential equation (16). The end points of the interval $[x_3, x_4]$ correspond to $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, respectively (see inequality (32)). The critical points correspond to $-\frac{\pi}{6}$ and $\frac{\pi}{6}$, respectively, and the solutions of the equation $f(x) = f(-\frac{b}{3a})$ are given in equation (23) and correspond to $-\frac{\pi}{3}$, 0, and $\frac{\pi}{3}$, respectively. This is summarized in figure 5.

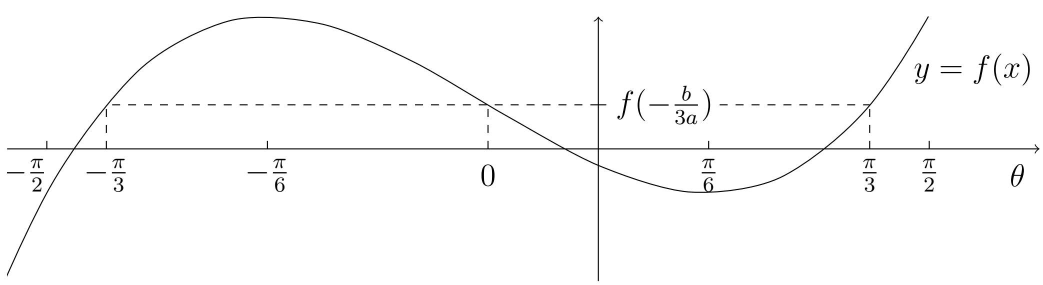


FIGURE 5. The sinusoidal zone parametrized using the sine function.

The solutions x_k, $k \in \{-1, 0, 1\}$, given in formulas (22) of the cubic equation can be expressed in terms of the angle θ. There is a solution corresponding to an angle between $-\pi/2$ and $-\pi/6$, a second solution for an angle between $-\pi/6$ and $\pi/6$, and a third solution for a angle between $\pi/6$ and $\pi/2$. The solution for the angle between $-\pi/2$ and $-\pi/6$ is expressed as a shift from the angle $-\pi/3$, the second solution is expressed as an angle between $-\pi/6$ and $\pi/6$, and the third solution between $\pi/6$ and $\pi/2$ is expressed as a shift from the angle $\pi/3$.

In order to gain a better understanding of the solutions of the equation $f(x) = 0$, in the case $\Delta_3(f) > 0$ let us expand $f(x)$ as a Taylor polynomial near the abcissa of the inflection point $x = -\frac{b}{3a}$, then

$$f(x) = f(-\frac{b}{3a}) + f'(-\frac{b}{3a})\left(x + \frac{b}{3a}\right) + \frac{f'''(-\frac{b}{3a})}{6}\left(x + \frac{b}{3a}\right)^3,$$

therefore using the parametrization (52), and the fact that $f'''(x) = 6a$, the right hand side can be rewritten as

$$= f(-\frac{b}{3a}) + f'(-\frac{b}{3a})\left(\frac{2\sqrt{b^2-3ac}}{3a}\sin\theta\right) + a\left(\frac{2\sqrt{b^2-3ac}}{3a}\sin\theta\right)^3.$$

Therefore, by equation (4) we can simplify this to

$$\begin{aligned} &= f(-\tfrac{b}{3a}) - \tfrac{2(b^2-3ac)^{3/2}}{9a^2}\sin\theta + \tfrac{8(b^2-3ac)^{3/2}}{27a^2}\sin^3\theta \\ &= f(-\tfrac{b}{3a}) - \tfrac{2(b^2-3ac)^{3/2}}{27a^2}\left(3\sin\theta - 4\sin^3\theta\right) \\ &= f(-\tfrac{b}{3a}) - \tfrac{2(b^2-3ac)^{3/2}}{27a^2}\sin(3\theta). \end{aligned}$$

We have proved that

$$f(x) = f(-\frac{b}{3a}) - \frac{2(b^2-3ac)^{3/2}}{27a^2}\sin(3\theta). \tag{53}$$

We can use this equation to solve the cubic equation $f(x) = 0$. In this case we can solve for $\sin(3\theta)$ by

$$\sin(3\theta) = \frac{27a^2 f(-\frac{b}{3a})}{2(b^2-3ac)^{3/2}} = \frac{f(-\frac{b}{3a})}{2a\delta^3}. \tag{54}$$

Using equation (3) this can be written as

$$\sin(3\theta) = \frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}},$$

which can be solved by the formula

$$3\theta = k\pi + (-1)^k \arcsin\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right),$$

where k is an integer. Therefore,

$$\theta = \frac{k\pi}{3} + \frac{(-1)^k}{3}\arcsin\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right). \tag{55}$$

Looking back we see that equation (28) states the equality of the angle expressed in the previous equation, with the angle that can be solved from the definition of angle θ in equation (52). Choosing $k = -1$, $k = 0$, and $k = 1$ in the previous equation gives us the solutions in the three intervals into which the sinusoidal zone is divided.

Equation (53) implies that the quantity $\frac{2(b^2-3ac)^{3/2}}{27a^2}$ represents the vertical distance between the ordinate of the inflection point of the graph and the relative local minimum and maximum of the graph. In terms of δ this distance can be computed as $2a\delta^3$. See figure 6.

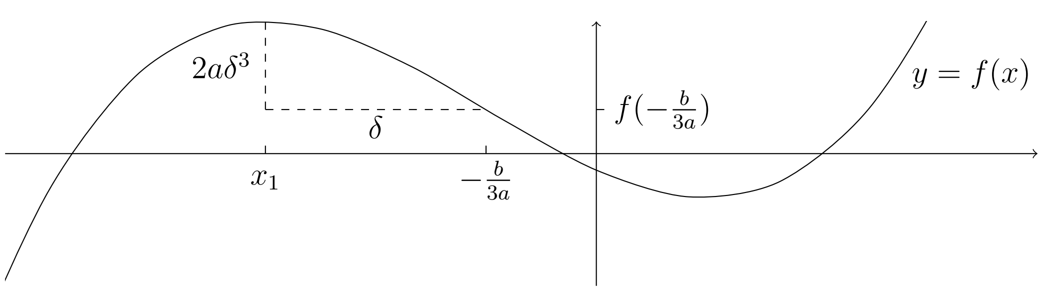


Figure 6. The vertical and horizontal displacement from the inflection point and a local maximum.

Now we are ready to explain how the angle 3θ that appears in equation (54) can be seen geometrically through an auxiliary construction in the graph of f. Construct a circle with center at the inflection point of the graph and radius $2a\delta^3$. Then this circle will have its highest point at the same level as the local maximum of f and its lowest point at the same level as the local minimum of f. Given that $\Delta_3(f) > 0$, the local maximum and minimum of f are in opposite sides of the x-axis (see equation (10)), therefore the circle we constructed must intersect the x-axis at two points, P and Q. See figure 7.

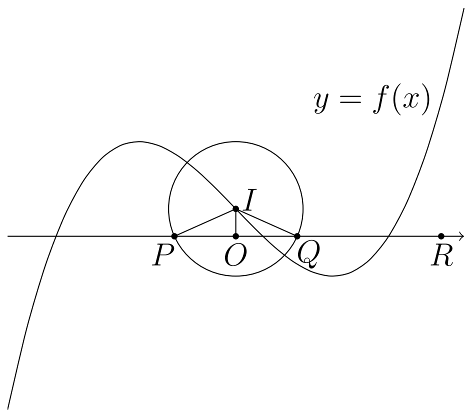


FIGURE 7. Circle with center at the inflection point I intersects the x-axis at P and Q.

In figure 7 we constructed two extra points O and R. Point O represents the orthogonal projection of I onto the x-axis, while R is a point in the x-axis that determines its positive direction. In the case that the inflection point is above the x-axis $OI = f(-\frac{b}{3a})$ and $IP = IQ = 2a\delta^3$, therefore

$$\sin(\angle IPO) = \frac{f(-\frac{b}{3a})}{2a\delta^3} = \sin(3\theta).$$

Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ the angle 3θ could represent $\angle IPR$ or $\angle IQR$, that is, the angles that the lines IP and IQ make with the positive direction of the x-axis, both angles measured counterclockwise, and the angle $\angle RQI$ measured clockwise. See figure 8.

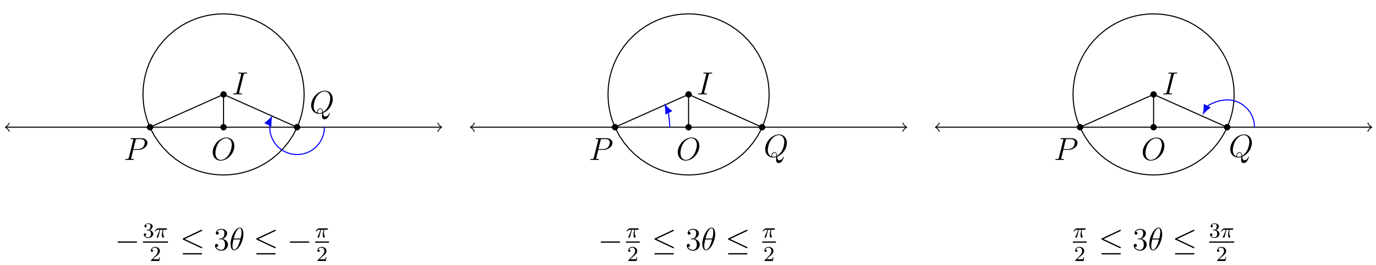


FIGURE 8. The angle 3θ that yields solutions to the cubic equation.

In the case that the inflection point is below the x-axis

In figure 9 we constructed two extra points O and R. Point O represents the orthogonal projection of I onto the x-axis, while R is a point in the x-axis that determines its positive direction. In the case that the inflection point is below the x-axis $OI = -f(-\frac{b}{3a})$ and $IP = IQ = 2a\delta^3$, therefore

$$\sin(\angle OPI) = \frac{-f(-\frac{b}{3a})}{2a\delta^3} = \sin(3\theta).$$

Here we must make the convention that we measure θ clockwise from the x-axis down. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ the angle 3θ could represent $\angle IPR$ or $\angle IQR$, that is, the angles that the lines IP and IQ make

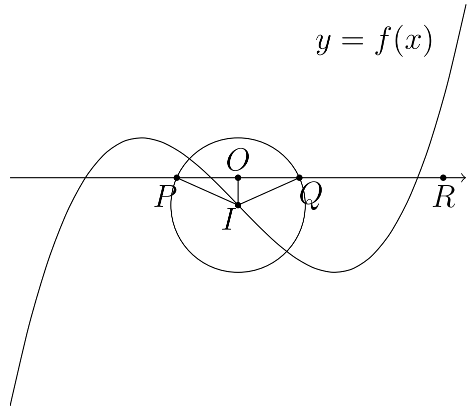
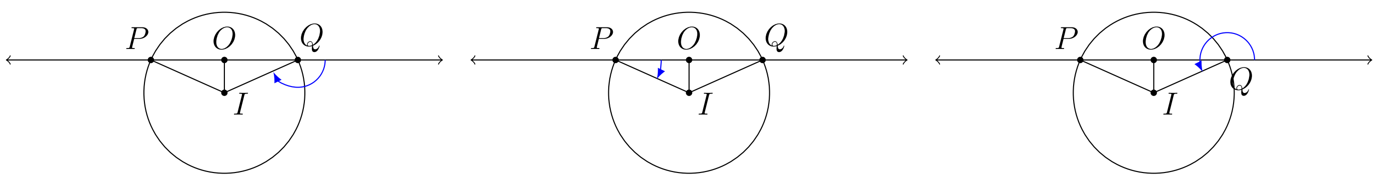


FIGURE 9. Circle with center at the inflection point I intersects the x-axis at P and Q.

with the positive direction of the x-axis, both angles measured clockwise, and the angle $\angle RQI$ measured counterclockwise. See figure 8.



$-\frac{3\pi}{2} \leq 3\theta \leq -\frac{\pi}{2}$ $\quad -\frac{\pi}{2} \leq 3\theta \leq \frac{\pi}{2}$ $\quad \frac{\pi}{2} \leq 3\theta \leq \frac{3\pi}{2}$

FIGURE 10. The angle 3θ that yields solutions to the cubic equation.

In order to simplify the writing of the formulas in the case $\Delta_3(f) > 0$ let us write

$$p = \frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}},$$

then the solutions s_k of the cubic equation can be written as

$$s_{-1} = -\frac{b}{3a} - \frac{\sqrt{b^2 - 3ac}}{3a}\left(\sin\left(\frac{\arcsin(p)}{3}\right) + \sqrt{3}\cos\left(\frac{\arcsin(p)}{3}\right)\right) \tag{56}$$

$$s_0 = -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a}\sin\left(\frac{\arcsin(p)}{3}\right) \tag{57}$$

$$s_1 = -\frac{b}{3a} - \frac{\sqrt{b^2 - 3ac}}{3a}\left(\sin\left(\frac{\arcsin(p)}{3}\right) - \sqrt{3}\cos\left(\frac{\arcsin(p)}{3}\right)\right). \tag{58}$$

which means that the solutions of the cubic equation can be written in the form

$$\begin{aligned} s_0 &= -\tfrac{b}{3a} + 2\rho \\ s_{-1} &= -\tfrac{b}{3a} - \rho - \xi \\ s_1 &= -\tfrac{b}{3a} - \rho + \xi. \end{aligned}$$

The solutions s_{-1} and s_1 can be written in the form

$$s = -\frac{b}{3a} - \frac{\sqrt{b^2 - 3ac}}{3a}\left(\sin\left(\frac{\arcsin(p)}{3}\right) \pm \sqrt{3}\cos\left(\frac{\arcsin(p)}{3}\right)\right)$$

We split these solutions into two terms

$$\begin{aligned} x &= -\frac{b}{3a} - \frac{\sqrt{b^2-3ac}}{3a}\sin\left(\frac{\arcsin(p)}{3}\right) \\ y &= \frac{\sqrt{3}\sqrt{b^2-3ac}}{3a}\cos\left(\frac{\arcsin(p)}{3}\right), \end{aligned}$$

and observe that the solutions s_{-1} and s_1 can be written from the numbers x and y above as $x \pm y$.

Observe that the points (x, y) and $(x, -y)$ are in the ellipse with equation

$$\left(x + \frac{b}{3a}\right)^2 + \frac{y^2}{3} = \frac{b^2 - 3ac}{9a^2}. \tag{59}$$

This is the equation of an ellipse centered at $(-\frac{b}{3a}, 0)$ whose foci are at $(-\frac{b}{3a}, x_3)$ and $(-\frac{b}{3a}, x_4)$.

In the figure below we can see a process that leads geometrically from the solution s_0 to the other two solutions s_{-1} and s_1. Let us call I the abcissa of the inflection point of the graph and let us construct an ellipse with horizontal minor axis $\delta = \frac{\sqrt{b^2-3ac}}{3a}$, and vertical major axis $\sqrt{3}\delta$. Let us reflect the solution s_0 across the abcissa I of the inflection point, so that I divides the segment from s_0 to x in the $2:1$ ratio. The vertical line through x intersects the ellipse at (x, y) and $(x, -y)$. The circle with center $(x, 0)$ and radius y intersects the x-axis at the solutions s_{-1} and s_1.

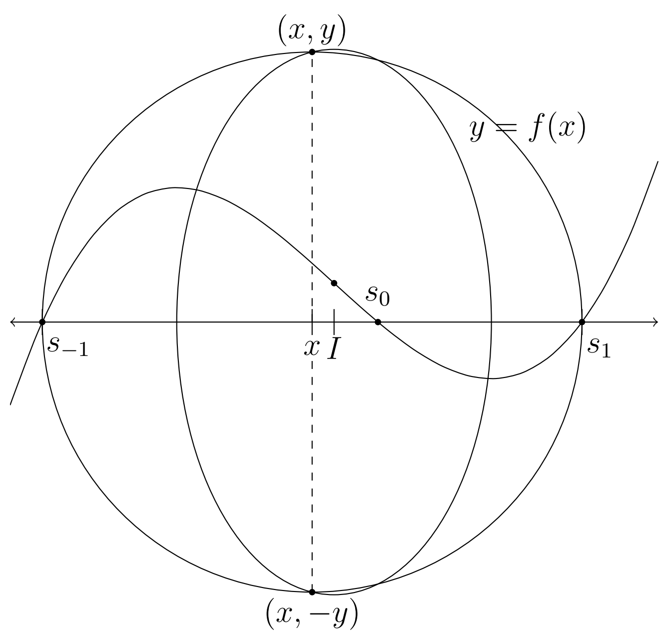


FIGURE 11. The solution s_0 determines s_{-1} and s_1.

Another important observation about formulas (56), (57), and (58) is the two different exponents of the term $b^2 - 3ac$ in these formulas. The exponent of the terms outside the trigonometric functions is lower than the one that appears inside the trigonometric functions. This can be understood by using the fact that

$$4\sin x \sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{\pi}{3} - x\right) = -\sin 3x,$$

for any angle x as follows. First notice that formulas (56), (57), and (58) imply that

$$\left(s_{-1} + \frac{b}{3a}\right)\left(s_0 + \frac{b}{3a}\right)\left(s_1 + \frac{b}{3a}\right) = \frac{2(b^2-3ac)^{3/2}}{27a^3}\sin\mu(\sin\mu + \sqrt{3}\cos\mu)(\sin\mu - \sqrt{3}\cos\mu),$$

where $\mu = \frac{\arcsin p}{3}$. This multiplication has raised the exponent of $b^2 - 3ac$ from $1/2$ to $3/2$ in the term outside the trigonometric functions. But by the triple angle identity above this is equal to

$$= -\frac{2(b^2-3ac)^{3/2}}{27a^3}\sin 3\mu = -\frac{2(b^2-3ac)^{3/2}}{27a^3}\sin\arcsin p = -\frac{2(b^2-3ac)^{3/2}}{27a^3}p.$$

Now we can see that since p has a denominator which contains the term $(b^2-3ac)^{3/2}$, that these terms coming from outside the sine function and inside the sine function will cancel each other out in the previous expression, and so the previous expression can be simplified to

$$-\frac{27a^2d + 2b^3 - 9abc}{27a^3} = -\frac{f\left(-\frac{b}{3a}\right)}{a}.$$

In other words, the reason why the exponents in $b^2 - 3ac$ are different outside of the sine term and inside the sine term is because only after multiplying those terms we can balance the amount in the outside to increase its exponent 3-fold to match the term inside the trigonometric sine function that is already cubed.

6. Complex Solutions of a Cubic Equation

Once a solution of an equation has been deduced we can use polynomial division to obtain a new equation for the remaining solutions. If $f(x) = ax^3+bx^2+cx+d$ and $f(\alpha) = 0$, then we can factor $f(x) = (x-\alpha)g(x)$ for some polynomial g.

Explicitly,

$$g(x) = \frac{f(x)}{x-\alpha} = \frac{f(x)-f(\alpha)}{x-\alpha} = ax^2 + (a\alpha+b)x + a\alpha^2 + b\alpha + c. \tag{60}$$

The other solutions of the cubic equation are the solutions of the equation

$$ax^2 + (a\alpha+b)x + a\alpha^2 + b\alpha + c = 0. \tag{61}$$

The discriminant of g is given by

$$\Delta_2(g) = (a\alpha+b)^2 - 4a(a\alpha^2+b\alpha+c) = b^2 - 4ac - 2ab\alpha - 3a^2\alpha^2 = \frac{4}{3}(b^2-3ac) - 3a^2\left(\alpha+\frac{b}{3a}\right)^2. \tag{62}$$

We will apply this formula, and later solve the equation $g(x) = 0$, for each of the real solutions we have found before.

First, we start with the formulas (33) and (34). These formulas have the form

$$\alpha = -\frac{b}{3a} \pm \frac{2\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}(\mp p)}{3}\right)$$

where

$$p = \frac{27a^2d + 2b^3 - 9ab}{2(b^2-3ac)^{3/2}}.$$

Therefore

$$\begin{aligned}
\Delta_2(g) &= \frac{4}{3}(b^2-3ac) - 3a^2\left(\alpha+\frac{b}{3a}\right)^2 \\
&= \frac{4}{3}(b^2-3ac) - \frac{4}{3}(b^2-3ac)\cosh^2\left(\frac{\cosh^{-1}(\mp p)}{3}\right) \\
&= -\frac{4}{3}(b^2-3ac)\sinh^2\left(\frac{\cosh^{-1}(\mp p)}{3}\right).
\end{aligned} \tag{63}$$

On the other hand,

$$a\alpha + b = \frac{2b}{3} \pm \frac{2\sqrt{b^2-3ac}}{3}\cosh\left(\frac{\cosh^{-1}(\mp p)}{3}\right). \tag{64}$$

This implies the following theorem, which completes Theorem 6.

Theorem 7. *Let a, b, c and d be real numbers. Let $f(x) = ax^3 + bx^2 + cx + d$. Assume that $a > 0$, $\Delta_3(f) < 0$, and $b^2 - 3ac > 0$, then there are two complex solutions of the equation $f(x) = 0$, given by*

$$x = -\frac{b}{3a} + \frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right)\right) \tag{65}$$

when $27a^2d + 2b^3 - 9abc > 0$, and

(66)

$$x = -\frac{b}{3a} - \frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}\left(-\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(-\frac{27a^2d+2b^3-9abc}{2(b^2-3ac)^{3/2}}\right)}{3}\right)\right)$$

when $27a^2d + 2b^3 - 9abc < 0$.

Proof. First let us deduce (65) from (33). By equations (64) and (63) we have

$$x = \frac{-\frac{2b}{3} + \frac{2\sqrt{b^2-3ac}}{3}\cosh\left(\frac{\cosh^{-1}(p)}{3}\right) \pm \frac{2\sqrt{3}}{3}i\sqrt{b^2-3ac}\sinh\left(\frac{\cosh^{-1}(p)}{3}\right)}{2a},$$

hence

$$\begin{aligned} x &= \frac{-b + \sqrt{b^2-3ac}\cosh\left(\frac{\cosh^{-1}(p)}{3}\right) \pm i\sqrt{3}\sqrt{b^2-3ac}\sinh\left(\frac{\cosh^{-1}(p)}{3}\right)}{3a} \\ &= -\frac{b}{3a} + \frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}(p)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}(p)}{3}\right)\right), \end{aligned}$$

from where equation (65) follows immediately. We follow the same procedure to deduce (66) from (34). This leads us to

$$x = -\frac{b}{3a} - \frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}(-p)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}(-p)}{3}\right)\right),$$

from where equation (66) follows immediately. □

Let us take a look at the formulas (65) and (66) for the complex solutions. We know they are conjugate of each other, and they have the form

$$\alpha = -\frac{b}{3a} + \lambda\frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right)\right),$$

where $\lambda = \operatorname{sign}(27a^2d + 2b^3 - 9abc)$. See foonote[2]. The real and imaginary parts are given by

$$\begin{aligned} x &= -\frac{b}{3a} + \lambda\frac{\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \\ y &= \pm i\lambda\frac{\sqrt{3}\sqrt{b^2-3ac}}{3a}\sinh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right). \end{aligned}$$

It follows that

$$\left(x + \frac{b}{3a}\right)^2 - \frac{y^2}{3} = \frac{b^2-3ac}{9a^2}. \tag{67}$$

[2]The sign function is defined as $\operatorname{sign}(x) = |x|/x$. It is equal to 1 when $x > 0$ and equal to -1 when $x < 0$. Observe that $|x| = x \operatorname{sign} x$.

This is a hyperbola with center $(-\frac{b}{3a}, 0)$. Their foci are located at $-\frac{b}{3a} \pm c$, where $c^2 = \frac{b^2-3ac}{9a^2} + \frac{3(b^2-3ac)}{9a^2} = \frac{4(b^2-3ac)}{9a^2}$. This means that the foci are located at $-\frac{b}{3a} \pm \frac{2\sqrt{b^2-3ac}}{3a}$, that is, at x_3 and x_4. The asymptotes of this hyperbola have equations

$$x + \frac{b}{3a} - \frac{y}{\sqrt{3}} = 0, \text{ and } x + \frac{b}{3a} + \frac{y}{\sqrt{3}} = 0,$$

which make angles of $\pi/3$ and $2\pi/3$ with the positive direction of the x-axis, respectively.

Under the hypothesis of Theorem 7 the two complex solutions of the cubic equation are in the hyperbola with equation (67). Solution (65) is in the right branch and solution (66) is in the left branch.

If we look at equations (65) and (66) we notice that the complex solutions we have found are never real, because their imaginary part is only zero when

$$\cosh^{-1}\left|\frac{27a^2d + 2b^3 - 9abc}{2(b^2-3ac)^{3/2}}\right| = 0,$$

and that is only possible when $|27a^2d + 2b^3 - 9abc| = 2(b^2 - 3ac)^{3/2}$, therefore

$$27a^2\Delta_3(f) = 4(b^2 - 3ac)^3 - (27a^2d + 2b^3 - 9abc)^2 = 0.$$

This implies that $\Delta_3(f) = 0$. This means that when d is chosen so that $|27a^2d+2b^3-9abc| = 2(b^2-3ac)^{3/2}$ then $\Delta_3(f) = 0$. Solving this, as an equation for d, yields two solutions d_1 and d_2 where $d_1 < d_2$.

It follows that the complex solutions converge to x_2 when d approaches d_2 from the right, in fact, if $d > d_2$, then $\lambda = \text{sign}(27a^2d_2 + 2b^3 - 9abc) = \text{sign}(2(b^2 - 3ac)^{3/2} = 1$, so that

$$\lim_{d\to d_2^+} \alpha = -\frac{b}{3a} + \frac{\sqrt{b^2-3ac}}{3a}\left(\cosh\frac{\cosh^{-1}(|1|)}{3} \pm i\sqrt{3}\sinh\frac{\cos^{-1}(|1|)}{3}\right) = -\frac{b}{3a} + \frac{\sqrt{b^2-3ac}}{3a} = x_2$$

therefore α approaches x_2 as d aproaches d_2 from the right. In particuluar when d takes the value d_2, the two complex solutions that exist for $d > d_2$ merge into one (because $\Delta_3(f) = 0$) and it happens at a critical point. In addition, the other solution of the equation converges to x_3, because the other solution of the equation is given by

$$\beta = -\frac{b}{3a} - \frac{2\lambda\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}}{3}\right)$$

which approaches

$$-\frac{b}{3a} - \frac{2\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}(1)}{3}\right) = -\frac{b}{3a} - \frac{2\sqrt{b^2-3ac}}{3a} = x_3.$$

as d approaches d_2 from the right.

In a similar way the complex solutions approach x_1 as d approaches d_1 from the left, while the real solution approaches x_4.

This leads to the following analysis of the behavior of the three solutions of the cubic equation in the case that $b^2 - 3ac > 0$ as d increases.

If $d < d_1$, then there is only one real solution bigger than x_4, while the other two complex solutions are symmetrically located in the left branch of the hyperbola (67). As we increase d the real solution moves towards x_4, while the complex solutions move towards the local maximum at x_1. When $d = d_1$ the two complex solutions merge at x_1, while the other solution reaches x_4. If we increase d from d_1 to d_2 we will enter the region where $\Delta_3(f) > 0$ and there will be three real solutions. The two solutions that merged at x_1 will split, one travelling to the left towards x_3 and the other to the right towards x_2, while the solution at x_4 will move to the left towards x_2. When d reaches d_2 the two real solutions that were travelling towards x_2 will merge there, while the solution that was travelling towards x_3 will reach x_3. Finally, when $d > d_2$ we will go back to the region where $\Delta_3(f) < 0$, and the solution that became x_3 will continue moving towards the left, while the solutions that merged at x_2 will move towards the complex plane to travel in the right branch of the hyperbola (67) symmetrically towards infinity.

Now we are ready to move to the case $b^2 - 3ac < 0$.

Theorem 8. *Let a, b, c and d be real numbers. Let $f(x) = ax^3 + bx^2 + cx + d$. Assume that $a > 0$ and $b^2 - 3ac < 0$, then there are two complex solutions of the equation $f(x) = 0$, given by*

$$x = -\frac{b}{3a} + \frac{\sqrt{3ac - b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right) \tag{68}$$

Proof. In this case we start with the solution (17)

$$\alpha = -\frac{b}{3a} - \frac{2\sqrt{3ac - b^2}}{3a}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d + 2b^3 - 9abc}{2(3ac - b^2)^{3/2}}\right)\right)$$

We compute the quadratic function $g(x)$ given in equation (60), and then we compute its discriminant using formula (62) and we obtain

$$\Delta_2(g) = -\frac{4}{3}(3ac - b^2)\cosh^2\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d + 2b^3 - 9abc}{2(3ac - b^2)^{3/2}}\right)\right)$$

Solving the quadratic equation $g(x) = 0$ using the quadratic formula leads to formulas (68). □

A summary of all the solutions of a cubic equation that has complex solutions is given in figure 12

Observe that if

$$\alpha = -\frac{b}{3a} + \frac{\sqrt{3ac - b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right)$$

are the complex solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$ in the case $a > 0$ and $b^2 - 3ac < 0$, then its real an imaginary parts are given by

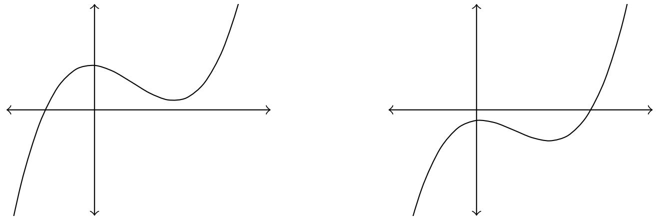
$$\begin{aligned} x &= -\frac{b}{3a} + \frac{\sqrt{3ac-b^2}}{3a}\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \\ y &= \frac{\sqrt{3}\sqrt{3ac-b^2}}{3a}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \end{aligned}$$

from where

$$\left(x + \frac{b}{3a}\right)^2 - \frac{y^2}{3} = -\frac{3ac - b^2}{9a^2}.$$

It follows that the solutions also satisfy equation (67). This also means that the asymptotes of this hyperbola are the same as the equations of the asymptotes in the case $b^2 - 3ac > 0$. The foci of this hyperbola are located at $(-\frac{b}{3a}, \pm\frac{2\sqrt{3ac-b^2}}{3a})$. Observe that because this hyperbola is oriented vertically, there is one solution in the upper branch and one solution in the lower branch.

The knowledge that the complex solutions are in the hyperbola with equation (67) allows us to find the complex solutions of the cubic equation geometrically. In fact, if α, β, and γ are the solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then $\alpha + \beta + \gamma = -\frac{b}{a}$, so the center of gravity of the triangle formed by the solutions is located at $\frac{\alpha+\beta+\gamma}{3} = -\frac{b}{3a}$, that is, the center of gravity of the triangle of the solutions is located at the center of the hyperbola (67). In particular, once a real solution is known, we can find the real part of the other solutions because the mid-point of the complex solutions is on the x-axis and the center of gravity of a triangle divides the segment joining a vertex with the mid-point of the opposite side in a 2 : 1 ratio, then the imaginary parts of these solutions can be found in the hyperbola (67).

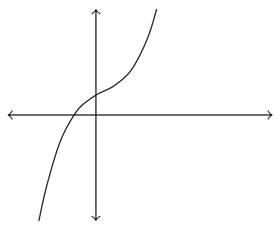


$a > 0$, $\Delta_3(f) < 0$, and $b^2 - 3ac > 0$.

$$x = -\frac{b}{3a} - \frac{2\lambda\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right),$$

$$x = -\frac{b}{3a} + \frac{\lambda\sqrt{b^2-3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right)\right),$$

where $\lambda = \operatorname{sign}(27a^2d + 2b^3 - 9abc)$.



$a > 0$ and $b^2 - 3ac < 0$.

$$x = -\frac{b}{3a} - \frac{2\sqrt{3ac-b^2}}{3a}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)\right),$$

$$x = -\frac{b}{3a} + \frac{\sqrt{3ac-b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right).$$

FIGURE 12. Formulas to solve the equation $ax^3 + bx^2 + cx + d = 0$ when there are complex number solutions.

7. Complex Solutions in Radical Form

Complex solutions to a cubic equation, written in radical form, can be obtained in the same way that we obtained the radical solutions to the cubic equation in section 4. In order to do that we need the following identities

$$\cosh\left(\frac{\sinh^{-1}(p)}{3}\right) = \frac{1}{2}\left(\sqrt[3]{\sqrt{p^2+1}+p} + \frac{1}{\sqrt[3]{\sqrt{p^2+1}+p}}\right), \tag{69}$$

and

$$\sinh\left(\frac{\cosh^{-1}(p)}{3}\right) = \frac{1}{2}\left(\sqrt[3]{\sqrt{p^2-1}+p} - \frac{1}{\sqrt[3]{\sqrt{p^2-1}+p}}\right), \tag{70}$$

for $p \geq 1$.

We deduce the following formulas to solve a cubic equation.

Theorem 9. *Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers and $a > 0$, and let ω be a primitive cube root of 1. If $b^2 - 3ac < 0$, then the solutions of the equation $f(x) = 0$ are*

$$x_k = -\frac{b}{3a} - \frac{\omega^k \sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(b^2 - 3ac)\omega^{2k}}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}, \tag{71}$$

where $k \in \{0, 1, 2\}$.

Proof. The case $k = 0$ is covered by formula (50). The formula for x_2 follows from x_1 by conjugation. The formula for x_2 is like the formula for x_1 in that the second term has the primitive root ω^2, while the third term has the square of this $\omega = (\omega^2)^2$. This makes the value of ω not important, but for purposes of this proof we will use $\omega = \frac{-1+i\sqrt{3}}{2}$, so that $\omega^2 = \frac{-1-i\sqrt{3}}{2}$.

In formula (68) we saw that we can take

$$x_1 = -\frac{b}{3a} + \frac{\sqrt{3ac - b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}(p)}{3}\right) - i\sqrt{3}\cosh\left(\frac{\sinh^{-1}(p)}{3}\right)\right)$$

where

$$p = \frac{27a^2d + 2b^3 - 9abc}{2(3ac - b^2)^{3/2}}.$$

Using identities (48) and (69) we can write x_1 as.

$$-\frac{b}{3a} + \frac{\sqrt{3ac - b^2}}{3a}\left(\frac{1}{2}\left(\sqrt[3]{\sqrt{p^2+1} + p} - \frac{1}{\sqrt[3]{\sqrt{p^2+1} + p}}\right) - \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\sqrt{p^2+1} + p} + \frac{1}{\sqrt[3]{\sqrt{p^2+1} + p}}\right)\right)$$

$$= -\frac{b}{3a} - \frac{\omega\sqrt{3ac - b^2}}{3a}\sqrt[3]{\sqrt{p^2+1} + p} - \frac{\omega^2\sqrt{3ac - b^2}}{3a\sqrt[3]{\sqrt{p^2+1} + p}}.$$

The proof is concluded by using equality (49) in the previous equation. □

The following theorem can be proved in a similar form to the previous theorem and the details are skipped.

Theorem 10. *Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers and $a > 0$, and let ω be a primitive cube root of 1. If $\Delta_3(f) < 0$ and $b^2 - 3ac > 0$, then the solutions of the equation $f(x) = 0$ are*

(72)

$$x_k = -\frac{b}{3a} + \frac{\lambda\omega^k \sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + |27a^2d + 2b^3 - 9abc|}}{3a\sqrt[3]{2}} + \frac{\lambda\sqrt[3]{2}(b^2 - 3ac)\omega^{2k}}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + |27a^2d + 2b^3 - 9abc|}},$$

where $k \in \{0, 1, 2\}$ and $\lambda = \operatorname{sign}(27a^2d + 2b^3 - 9abc)$.

8. General Formulas for Solving the Cubic Equation with Real Coefficients

Throughout our discussion we have assumed that $a > 0$ as a way to simplify and canonicalize our analysis. This hypothesis allowed us to draw conclusions such as $x_3 < x_1 < -\frac{b}{3a} < x_2 < x_4$, and these inequalities would reverse in the case that $a < 0$. We will show here that the formulas that we deduced earlier are general and apply to every problem.

Theorem 11. *Assume that a, b, c and d are real numbers such that $a \neq 0$, let $\lambda = \operatorname{sign}(27a^2d + 2b^3 - 9abc)$, then the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ are the following:*

If $\Delta_3(f) > 0$ there are three real solutions given by

$$x_k = -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a}\sin\left(\frac{k\pi}{3} + \frac{(-1)^k}{3}\arcsin\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right) \tag{73}$$

where $k \in \{-1, 0, 1\}$, and $ax_{-1} < ax_0 < ax_1$.

If $b^2 - 3ac < 0$, there is a real number solution given by

$$x = -\frac{b}{3a} - \frac{2\sqrt{3ac - b^2}}{3a} \sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d + 2b^3 - 9abc}{2(3ac - b^2)^{3/2}}\right)\right) \tag{74}$$

and two complex solutions that can be written in terms of hyperbolic functions as

$$x_{\pm} = -\frac{b}{3a} + \frac{\sqrt{3ac - b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right), \tag{75}$$

and can be written in terms of radicals as

$$x_k = -\frac{b}{3a} - \frac{\omega^k \sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(b^2 - 3ac)\omega^{2k}}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc}}, \tag{76}$$

where $k = 1$ or $k = 2$.

If $\Delta_3(f) < 0$ and $b^2 - 3ac > 0$, the real solution is given by

$$x = -\frac{b}{3a} - \frac{2\lambda\sqrt{b^2 - 3ac}}{3a}\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{|27a^2d + 2b^3 - 9abc|}{2(b^2 - 3ac)^{3/2}}\right)\right) \tag{77}$$

and the complex solutions can be written in terms of hyperbolic function as,

$$x = -\frac{b}{3a} + \lambda\frac{\sqrt{b^2 - 3ac}}{3a}\left(\cosh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)}{3}\right)\right), \tag{78}$$

and can be written in terms of radicals as

$$x_k = -\frac{b}{3a} + \frac{\lambda\omega^k \sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + |27a^2d + 2b^3 - 9abc|}}{3a\sqrt[3]{2}} + \frac{\lambda\sqrt[3]{2}(b^2 - 3ac)\omega^{2k}}{3a\sqrt[3]{\sqrt{-27a^2\Delta_3(f)} + |27a^2d + 2b^3 - 9abc|}}, \tag{79}$$

where $k = 1$ or $k = 2$.

Proof. A standard way to transform the general case to the case $a > 0$ is by *dividing* the equation by a, so that the leading coefficient will be equal to 1. However, another technique to transform the equation to one with a positive leading coefficient is by *multiplying* it by a. We analyze each formula when we input coefficients μa, μb, μc, and μd, where $\mu = a$. Observe that quantities such as $\Delta_3(f)$ and $b^2 - 3ac$ do not change signs when we switch the coefficients from a, b, c, d to μa, μb, μc, μd, because these quantities are homogeneous of even degree. Let us write $m = \operatorname{sign} a$.

We start analyzing the case $\Delta_3(f) > 0$. In this case the homogeneity of the terms in formula (22) produces the solutions

$$x_k = -\frac{b}{3a} + \frac{2m\sqrt{b^2 - 3ac}}{3a}\sin\left(\frac{k\pi}{3} + \frac{(-1)^k}{3}\arcsin\left(m^3\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right)$$

Now since $m = 1$ or $m = -1$, then $m^3 = m$. Additionally, $\sin mx = m\sin x$, for all x, and $\arcsin mx = m\arcsin x$ for all x such that $|x| \leq 1$, and $(-1)^k = (-1)^{mk}$, so that we can rewrite the above equation as

$$x_k = -\frac{b}{3a} + \frac{2\sqrt{b^2 - 3ac}}{3a}\sin\left(\frac{mk\pi}{3} + \frac{(-1)^{mk}}{3}\arcsin\left(\frac{27a^2d + 2b^3 - 9abc}{2(b^2 - 3ac)^{3/2}}\right)\right)$$

This shows that $x_k(\mu a, \mu b, \mu c, \mu d) = x_{mk}(a, b, c, d)$, from where it follows that when $a < 0$ the solution we obtain by applying the formula to obtain x_1 (the biggest solution) produces x_{-1} (the smallest solution), and viceversa. The solution x_0 is always the middle solution, regardless of the sign of a.

In the case that $b^2 - 3ac < 0$, the real solution with coefficients μa, μb, μc, and μd can be written according to formula (17) as

$$x = -\frac{b}{3a} - \frac{2m\sqrt{3ac-b^2}}{3a}\sinh\left(\frac{1}{3}\sinh^{-1}\left(m\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)\right),$$

but since $\sinh mx = m\sinh x$ and $\sinh^{-1} mx = m\sinh^{-1} x$ for all x, the above formula can be simplified to

$$x = -\frac{b}{3a} - \frac{2\sqrt{3ac-b^2}}{3a}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)\right).$$

This means that $x(\mu a, \mu b, \mu, \mu d) = x(a, b, c, d)$, so the solution is not affected by multiplying by a, and therefore is valid regardless of the sign of a.

In the case of the complex solutions, we use formula (68) with coefficients μa, μb, μc, and μd, and we obtain

$$x = -\frac{b}{3a} + \frac{m\sqrt{3ac-b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(m^3\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(m^3\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right),$$

Using that $m^3 = m \sinh mx = m\sinh x$, $\sinh^{-1} mx = m\sinh^{-1} mx$, and $\cosh mx = \cosh x$ for all x, the above formula can be simplified to

$$x_\pm = -\frac{b}{3a} + \frac{\sqrt{3ac-b^2}}{3a}\left(\sinh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right) \pm im\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(\frac{27a^2d+2b^3-9abc}{2(3ac-b^2)^{3/2}}\right)}{3}\right)\right).$$

In particular, if $a < 0$, then $x_+(\mu a, \mu b, \mu c, \mu d) = x_-(a, b, c, d)$ and $x_-(\mu a, \mu b, \mu c, \mu d) = x_+(a, b, c, d)$, so the formulas above applied when a is negative produce the solution but the $+$ solution yields the solution with the $-$ in the imaginary part, while the solution with the $-$ sign produces the solution with the $+$ in the imaginary part.

Given that the formulas for the radical solutions for a cubic equation were derived from the formulas using hyperbolic functions, the same statements about the way that the complex solutions switch are true for the formulas with radicals, that is, if $a < 0$, then $x_1(a, b, c, d) = x_2(\mu a, \mu b, \mu c, \mu d)$ and viceversa, $x_2(a, b, c, d) = x_1(\mu a, \mu b, \mu c, \mu d)$. In order to see this directly, we notice that replacing a, b, c, and d by μa, μb, μc, and μd in formula (76) yields

$$x_k = -\frac{b}{3a} - \frac{\omega^k\sqrt[3]{-\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(b^2-3ac)\omega^{2k}}{3a\sqrt[3]{-\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc}}.$$

From the perspective of the formula we want to get we need to find a way to switch the $-$ in front of the radical in $-\sqrt{-27a^2\Delta_3(f)}$ to a $+$ and leave it only as $\sqrt{-27a^2\Delta_3(f)}$ with no $-$ sign in front. This can be done by using identity (2), which implies that

$$-\sqrt{-27a^2\Delta_3(f)} + 27a^2d + 2b^3 - 9abc = \frac{4(b^2-3ac)^3}{\sqrt{-27a^2\Delta_3(f)}+27a^2d+2b^3-9abc},$$

which is enough to conclude the proof in this case.

Finally, we analyze the case $\Delta_3(f) < 0$ and $b^2 - 3ac > 0$. In this case formula (77) to find the real root will not change under the transformation from a, b, c, d to μa, μb, μc, μd, because we will get

$$x = -\frac{b}{3a} - \frac{2m^4\lambda\sqrt{b^2-3ac}}{3a}\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{|27a^2d+2b^3-9abc|}{2(b^2-3ac)^{3/2}}\right)\right)$$

The fourth power in m comes from two places. One m comes from the square root of $b^2 - 3ac$, which is later divided by a, and the other from λ, which contributes a m^3. Since $m^4 = 1$ this produces the same formula. The analysis for the complex solutions is done in the same way and this time there is no change between solutions, that is, x_1 remains x_1 and does not switch to x_2, and so does x_2. The same comment applies to the solutions given by radicals. □

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